Stochastic orderings on partial sums of random variables and on counting measures of random intervals

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Abstract

Relationships are established between the stochastic ordering of random variables, of their random partial sums, and of counting measures of random intervals. Two types of results are obtained. First, conditions are given under which a stochastic ordering between sequences of random variables is inherited by (vectors of) random partial sums of these variables. These results extend and generalize theorems of Borovkov, Ross and Stoyan. Second, conditions are provided under which the counting measures of a given point process in two ordered random intervals are also ordered.

These results are applied to several comparison problems in queueing systems. It is shown that if the service times in two $M/GI/1$ systems compare for some stochastic ordering, the busy periods compare for the same ordering. Various bounds on waiting times and response times are provided for queues with bulk arrivals. The cyclic and Bernoulli policies for customer allocation to parallel queues are compared in the transient regime. Stochastic ordering relations are established for the cycle times in polling systems.

Keywords: Partial Sums, Counting Measure, Stochastic Ordering, Stochastic Bounds, Busy Period, Bulk Arrival, Routing, Polling System.

Ordonnancement Stochastique de Sommes Partielles de Variables Aléatoires et de Mesures de Comptage d' Intervales Aléatoires

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Résumé

Nous étudions les ordonnancements stochastiques existant entre les sommes partielles aléatoires de variables aléatoires et entre les mesures de comptage d'intervalles aléatoires. Deux types de résultats sont présentés. D'une part, nous donnons des conditions sous lesquelles un ordre stochastique entre deux suites de variables aléatoires se transmet aux (vecteurs de) sommes partielles de ces variables. Ces résultats étendent et généralisent des théorèmes de Borovkov, Ross et Stoyan. D'autre part, nous obtenons des conditions sous lesquelles les nombres de points d'un processus ponctuel qui tombent dans des intervalles aléatoires de longueurs stochastiquement ordonnées sont également ordonnés.

Ces résultats sont ensuite appliqués à plusieurs problèmes de comparaison stochastique en théorie des files d'attente. Nous prouvons que deux files $M/GI/1$ dont les distributions des temps de service sont stochastiquement ordonnées ont des périodes d'activité ordonnées pour le même ordre. Nous obtenons également diverses bornes pour les files avec arrivés groupées. Dans le problème de l'allocation des clients à des files en parallèle, nous comparons la politique aléatoire ("Bernoulli") et la politique cyclique, dans le régime transitoire. Enfin, nous établissons des résultats d'ordonnancement stochastique pour les temps de cycle dans un système de "polling".

Mots Clef: Sommes Partielles, Mesures de Comptage, Ordonnancement Stochastique, Bornes Stochastiques, Période d'Activité, Arrivées Groupées, Routage, Polling.

1 Introduction

In this paper we address two related problems: the stochastic ordering of random partial sums and the stochastic ordering of random counting measures of some point process associated with random intervals.

In the first part, we study the relationship between the stochastic orderings of random variables and those of (vectors of) their random partial sums. We provide conditions under which a stochastic ordering between sequences of random variables is inherited by (vectors constructed with) random partial sums of these sequences.

The problem of comparing random partial sums of random variables arises in the study of stochastic systems with random dynamics, such as queues with bulk arrivals, vacation models or polling systems. It has been considered by several authors: Borovkov [7], Ross [18] and Stoyan [19]. Our results generalize theirs by weakening the statistical assumptions and by introducing other stochastic orderings. We also correct a mistake in the results of Stoyan. We then extend the ordering relationships to vectors of random partial sums.

In the second part, we compare the number of occurrences of a point process (i.e. the number of "arrivals") in two random intervals the lengths of which are stochastically ordered. This is, in some sense, a problem dual to the previous one.

The problem of comparing stochastically point processes has been studied by several authors, for instance Brown [9], Whitt [22], Lindvall [13], and Rolski and Szeckli [17]. Our problem differs from theirs in that we fix the point process and take two random intervals whereas in [9,13,17], roughly speaking, the authors fix the interval and compare the counting measures of two point processes.

It turns out that the structure of the point process dictates the way the ordering between the intervals is inherited by the numbers of occurrences. Any point process preserves the strong stochastic ordering, and any stationary point process preserves the convex ordering. The situation is more complex for pure renewal (Palm) processes, for which the preservation of the increasing convex ordering depends on the distribution of the interarrivals, and more precisely on the convexity of the renewal function. Processes with Gamma interarrivals are analyzed in detail and classified.

We apply these results in five examples of queueing systems. Some extend and generalize previous results and others are original.

The first example shows that in two $M/GI/1$ queues for which service times are ordered, the busy periods are ordered with the same ordering relation.

The second one consists in comparing single server queues with bulk arrivals: the $G^X/G/1$ model. We assume that interarrivals and service times are non-stationary. We study the bulk waiting times and construct for them the best possible upper and lower bounds in the sense of increasing convex ordering, among those that use $GI^X/GI/1$ queues (i.e. with stationary interarrivals and service times). The third example focuses on the $GI^X/GI/1$ model. A lower bound in the sense of the increasing convex ordering is obtained for the stationary customer response time by taking a deterministic bulk size.

The fourth application is concerned with the problem of customer allocation to parallel queues. For such systems, it is known [15] that the cyclic (Round Robin) allocation policy yields a smaller stationary customer waiting time (in the sense of the increasing convex ordering) than the Bernoulli (random) policy. This result is extended to the transient regime.

The last example deals with the comparison of polling systems. In a globally gated polling system, if the service times and/or the walking times are comparable for some ordering, then the cycle times are ordered for the same ordering relation.

The paper is organized as follows. In Section 2, we present the notation and some preliminaries on stochastic ordering. In Section 3, we analyze the relationship between the stochastic orderings of random variables and those of their random partial sums. In Section 4, we study the conditions under which the vectors of random partial sums are ordered. In Section 5, we present results on the inheritance of the stochastic ordering between random intervals by the number of occurrences of point processes in these intervals. The applications are presented in Section 6.

2 Notation and Preliminaries on Stochastic Ordering

We introduce here the notation for stochastic orderings and recall their basic properties. The reader is referred to [19] for more details and proofs. Throughout this paper, increasingness and positivity are understood to be non-strict, except when stated otherwise.

We begin with the definition of integral stochastic orderings. Let n be an arbitrary (strictly) positive integer. Let $C_{\mathcal{L}}$ be a class of functions from $\mathbb{R}^n$ to $\mathbb{R}$. Let $\boldsymbol{X}, \boldsymbol{Y} \in \mathbb{R}^n$ be two random vectors.

Definition 2.1 (Stochastic Ordering) *The random vector $\boldsymbol{X}$ is said to be smaller than the random vector $\boldsymbol{Y}$ in the sense of $\leq_{\mathcal{L}}$ (noted $\boldsymbol{X} \leq_{\mathcal{L}} \boldsymbol{Y}$), if $\forall f \in C_{\mathcal{L}}$, $E[f(\boldsymbol{X})] \leq E[f(\boldsymbol{Y})]$, provided that the expectations exist. The binary relation $\leq_{\mathcal{L}}$ is called the integral stochastic ordering, or simply the stochastic ordering, generated by the class of functions $C_{\mathcal{L}}$.*

A binary relation $\leq_{\mathcal{L}}$ thus defined is known to realize a partial preordering on the space of real distribution functions (or random variables) on $\mathbb{R}^n$. We shall often use "ordering" instead of "preordering".

Definition 2.2 *Let C^n_{st}, C^n_{cx}, C^n_{cv}, C^n_{icx}, C^n_{icv}, be respectively the class of all increasing, convex, concave, increasing and convex, increasing and concave functions from $\mathbb{R}^n$ into $\mathbb{R}$. These classes generate respectively the strong stochastic ordering ($\leq_{\mathrm{st}}$), the convex ordering ($\leq_{\mathrm{cx}}$), the concave ordering ($\leq_{\mathrm{cv}}$), the increasing convex ordering ($\leq_{\mathrm{icx}}$), and the increasing concave ordering ($\leq_{\mathrm{icv}}$).*

These five relations are actually partial orders. Other examples include the class of Schur convex functions (see e.g. [16]), the class of convex symmetric functions, the class "increasing product" functions, that is: $f(x_1, \ldots, x_n) = \prod_1^n f_i(x_i)$, where for all i, $f_i : \mathbb{R} \to \mathbb{R}$ is increasing. When $f_i(x_i)$ is a polynomial with positive coefficients, this class is called the class of "moment functions", noted $\mathcal{C}_\mathrm{m}^n$. One may also consider the class of linear functions $\mathcal{C}_\mathrm{l}^n$, for which the relation $X \leq_\mathrm{l} Y$ reduces to $EX \leq EY$.

We shall sometimes omit the superscript n. The dimension of the origin space will always be clear from the context. From the definition, it can be easily seen that:

Lemma 2.3 *We have the following equivalences:*

i) $X \leq_{\mathrm{cx}} Y \Leftrightarrow -X \leq_{\mathrm{cx}} -Y$

ii) $X \leq_{\mathrm{cx}} Y \Leftrightarrow X \geq_{\mathrm{cv}} Y$

iii) $X \leq_{\mathrm{cx}} Y \Leftrightarrow -X \geq_{\mathrm{cv}} -Y$

iv) $X \leq_{\mathrm{icx}} Y \Leftrightarrow -X \geq_{\mathrm{icv}} -Y$

v) $X \leq_{\mathrm{st}} Y \Leftrightarrow -X \geq_{\mathrm{st}} -Y$.

Lemma 2.4 *The following implication relations between the stochastic orderings hold:*

i) $\leq_{\mathrm{st}} \Rightarrow \leq_{\mathrm{icx}}$,

ii) $\leq_{\mathrm{cx}} \Rightarrow \leq_{\mathrm{icx}}$.

iii) $\leq_{\mathrm{cv}} \Rightarrow \leq_{\mathrm{icv}}$.

We shall also need the following analytical characterization of the increasing convex and increasing concave orderings:

Lemma 2.5 *Let $X, Y \in \mathbb{R}$ be two random variables (RV's).*

- $X \leq_{\mathrm{icx}} Y$ *if and only if for all $a \in \mathbb{R}$, $E[X-a]^+ \leq E[Y-a]^+$,*
- $X \leq_{\mathrm{icv}} Y$ *if and only if for all $a \in \mathbb{R}$, $E[a-X]^+ \geq E[a-Y]^+$,*

where $[x]^+ = \max(0, x)$.

Definition 2.6 *Let $\leq_\mathcal{C}$ be a stochastic ordering. The sequence of random variables $\{X_n\}_1^\infty$ is increasing (resp. decreasing) in the sense of $\leq_\mathcal{C}$, denoted by $\{X_n\} \uparrow \mathcal{C}$ (resp. $\{X_n\} \downarrow \mathcal{C}$) if for all $n \geq 1$, $X_n \leq_\mathcal{C} X_{n+1}$ (resp. $X_{n+1} \leq_\mathcal{C} X_n$).*

In what follows, $=_d$ denotes equality in distribution. The following two theorems are due to Strassen [20]:

Proposition 2.7 *Two random vectors $X, Y \in \mathbb{R}^n$ satisfy $X \leq_{st} Y$ if and only if there exist two random vectors X' and Y' defined on a common probability space, such that $X =_d X'$, $Y =_d Y'$, and $X' \leq Y'$ componentwise almost surely (a.s.).*

Proposition 2.8 *Two random variables $X, Y \in \mathbb{R}$ satisfy $X \leq_{cx} Y$ (resp. $X \leq_{icx} Y$) if and only if there exist two random variables X' and Y' defined on a common probability space such that $X =_d X'$, $Y =_d Y'$, and $X' = E[Y'|X']$ (resp. $X' \leq E[Y'|X']$) a.s..*

Corollary 2.9 *Two random variables $X, Y \in \mathbb{R}$ satisfy $X \leq_{cx} Y$ (resp. $X \leq_{cv} Y$) if and only if $X \leq_{icx} Y$ (resp. $X \leq_{icv} Y$) and $EX = EY$.*

Definition 2.10 (Convolution property) *The stochastic ordering $\leq_{\mathcal{L}}$ on random variables of $\mathbb{R}^n$ is said to satisfy the convolution property if for all random variables $X_1, X_2, Y_1, Y_2 \in \mathbb{R}^n$, with X_1 independent of X_2 and Y_1 independent of Y_2, the relations $X_1 \leq_{\mathcal{L}} Y_1$ and $X_2 \leq_{\mathcal{L}} Y_2$ imply $X_1 + X_2 \leq_{\mathcal{L}} Y_1 + Y_2$.*

Definition 2.11 (Stability with respect to translations) *A class C^n of functions is stable with respect to translations if for all $a \in \mathbb{R}^n$, $f \in C^n$ implies $f_a : x \mapsto f(x+a) \in C^n$.*

It is easy to see that the stability with respect to translations implies the convolution property (cf. [19, Proposition 1.1.2]):

Lemma 2.12 *Let $\leq_{\mathcal{L}}$ be the stochastic ordering generated by the class $C_{\mathcal{L}}$. If $C_{\mathcal{L}}$ is stable with respect to translations, then $\leq_{\mathcal{L}}$ has the convolution property.*

Consequently, the orderings $\leq_{st}, \leq_{cx}, \leq_{cv}, \leq_{icx}, \leq_{icv}$ satisfy the convolution property.

When the RV's are defined on $\mathbb{R}^{+n}$, the stability with respect to positive translations implies the convolution property. Thus, the moment ordering $\leq_m$ on positive RV's has also the convolution property.

3 Random Partial Sums

In this section, we consider the relationships between the stochastic orderings on the random variables and those on their random partial sums.

3.1 Description of the Problem

All random variables used in this section are defined on a common probability space. Let

- $\leq_{\mathcal{L}_1}, \leq_{\mathcal{L}_2}, \leq_{\mathcal{L}_3}$ be integral stochastic orderings satisfying the convolution property.
- A and B be two positive integer valued RV's,
- $\{X_n\}_1^\infty$ and $\{Y_n\}_1^\infty$ be two sequences of real valued RV's, mutually independent in each of the sequences,
- $U = \sum_1^A X_i$ and $V = \sum_1^B Y_i$. By convention, $U = 0$ if $A = 0$ and likewise for V.

It is assumed that

- $A \leq_{\mathcal{L}_1} B$,
- $X_n \leq_{\mathcal{L}_2} Y_n$ for all $n \geq 1$,
- the sets of RV's $\{A, B\}$ and $\{\{X_n\}, \{Y_n\}\}$ are mutually independent.

The question is:

> Under which conditions on the sequences $\{X_n\}$ and $\{Y_n\}$ is there an ordering relation between U and V: $U \leq_{\mathcal{L}_3} V$ for some $\leq_{\mathcal{L}_3}$?

This problem arises in the study of some stochastic systems, for instance the $GI^X/GI/1$ model (see Borovkov [7]) and branching processes (see Ross [18]). Other applications include polling systems and vacation models [21]. The most general results are stated by Stoyan [19, Proposition 2.2.5, p. 45], (see also Borovkov [7, Proposition 4, p. 143] and Ross [18, Lemma 8.6.7, p. 278]). In our notation, they write as:

Proposition 3.1 *Assume that $\{X_n\}$ and $\{Y_n\}$ are sequences of independent positive random variables. Let $\leq_{\mathcal{L}} \in \{\leq_{\text{icx}}, \leq_{\text{st}}\}$. If $\{X_n\} \uparrow \mathcal{L}$ and $\{Y_n\} \uparrow \mathcal{L}$, and if $\leq_{\mathcal{L}_1} = \leq_{\mathcal{L}_2} = \leq_{\mathcal{L}}$, then $\leq_{\mathcal{L}_3} = \leq_{\mathcal{L}}$.*

It turns out that the assumptions can be somewhat relaxed and that results are also available for other ordering relations.

3.2 Main Results

In this subsection, we present the results on the relationships between $\leq_{\mathcal{L}_3}$ and $\leq_{\mathcal{L}_1}, \leq_{\mathcal{L}_2}$, and leave their proofs to Section 3.3. We first present results for the most used orderings, and then propose a generalization.

Proposition 3.2 *Assume* $\leq_{\mathcal{L}_1}=\leq_{\rm st}$ *and* $\leq_{\mathcal{L}_2}=\leq_{\rm st}$. *If* $Y_n \geq 0$ *a.s. for* $n \geq 1$, *then* $\leq_{\mathcal{L}_3}=\leq_{\rm st}$.

Note that one does need the condition $Y_n \geq 0$ *a.s.* in order that $\leq_{\mathcal{L}_3}=\leq_{\rm st}$. It is not sufficient that $EY_n \geq 0$. For example: if $A = 1, B = 2$ a.s. and $\{X_n\}$, $\{Y_n\}$ i.i.d. with $X_n =_d Y_n = (-1,1)$ with probability $(1/2,1/2)$. Then $U = X$ and $V = (-2,0,2)$ with probabilities $(1/4,1/2,1/4)$ so that neither $X \leq_{\rm st} Y$ nor $Y \leq_{\rm st} X$.

However, the stochastic increasingness of the sequences $\{X_n\}$ and $\{Y_n\}$, required in Proposition 3.1 is not necessary for the $\leq_{\rm st}$ ordering.

Proposition 3.3 *Assume* $\leq_{\mathcal{L}_1}=\leq_{\rm st}$ *and* $\leq_{\mathcal{L}_2}=\leq_{\rm icx}$. *If* $EX_n \geq 0$ *or* $EY_n \geq 0$ *for* $n \geq 1$, *then* $\leq_{\mathcal{L}_3}=\leq_{\rm icx}$.

The assumption that $EY_n \geq 0$ is necessary for $\leq_{\mathcal{L}_3}=\leq_{\rm icx}$. Otherwise, take in particular $\{X_n\}$ and $\{Y_n\}$ to be i.i.d. and $X_n =_d Y_n$. If now $EA \leq EB$, Wald's Lemma implies: $(EX = EY < 0) \Rightarrow (EU = EA.EX \geq EB.EY = EV)$.

Proposition 3.4 *Assume* $\leq_{\mathcal{L}_1}=\leq_{\rm icx}$ *and* $\leq_{\mathcal{L}_2}=\leq_{\rm icx}$. *If either* $X_n \geq 0$ *a.s. for* $n \geq 1$ *and* $\{X_n\} \uparrow$ icx, *or* $Y_n \geq 0$ *a.s. for* $n \geq 1$ *and* $\{Y_n\} \uparrow$ icx, *then* $\leq_{\mathcal{L}_3}=\leq_{\rm icx}$.

One may have observed that Proposition 3.4 slightly relaxes the assumptions on the stochastic increasingness of the sequences $\{X_n\}$ and $\{Y_n\}$ in Proposition 3.1 for the $\leq_{\rm icx}$ ordering.

The positivity of $\{X_n\}$ or $\{Y_n\}$ assumed in Proposition 3.4 is necessary. Otherwise, one has the following counterexample (cf. [7, p. 144]): $A = 1$ a.s., $B = (0,2)$ with respective probabilities $(1/2,1/2)$ and $\{X_n\}$, $\{Y_n\}$ i.i.d. with $X_n =_d Y_n = (-1,1)$, also with probabilities $(1/2,1/2)$. With the function $f(x) = [x]^+$, one gets: $Ef(U) = 1/2 > 1/4 = Ef(V)$.

Proposition 3.5 *Assume* $\leq_{\mathcal{L}_1}=\leq_{\rm icv}$ *and* $\leq_{\mathcal{L}_2}=\leq_{\rm icv}$. *If either* $X_n \geq 0$ *a.s. for* $n \geq 1$ *and* $\{X_n\} \downarrow$ icv, *or* $Y_n \geq 0$ *a.s. for* $n \geq 1$ *and* $\{Y_n\} \downarrow$ icv, *then* $\leq_{\mathcal{L}_3}=\leq_{\rm icv}$.

Proposition 3.5 also appears in Stoyan's Proposition 2.2.5 [19, p. 45] but is misstated with "$\{X_n\} \uparrow$ icv". The result does not hold with $\{X_n\} \uparrow$ icv: if it were, then with Wald's Lemma, one would have in particular: $(A \leq_{\rm cv} B$, $\{X_n\} \uparrow$ cv, $\{Y_n\} \uparrow$ cv$) \Rightarrow (U \leq_{\rm cv} V)$, which, using Lemma 2.3, is equivalent to: $(B \leq_{\rm cx} A$, $\{X_n\} \downarrow$ cx, $\{Y_n\} \downarrow$ cx$) \Rightarrow (V \leq_{\rm cx} U)$. This is not compatible with Proposition 3.6 below.

Indeed, consider the following counterexample: let A take the values 0 or 2 with probabilities $1/2, 1/2$. Let $B = 1$ a.s. Take $\{X_n\} \equiv \{Y_n\}$. Let X_1 be a RV with mean 1 and second moment κ, and $X_n = X_2 = 1$ a.s. for $n \geq 2$. Then $EU^2 = (3+\kappa)/2$ and $EV^2 = \kappa$. Therefore, if $\kappa > 3$, $EV^2 > EU^2$ and $V \leq_{\rm cx} U$ is impossible.

Proposition 3.6 *Assume* $\leq_{\mathcal{L}_1}=\leq_{\mathrm{cx}}$ *and* $\leq_{\mathcal{L}_2}=\leq_{\mathrm{cx}}$. *If either* $X_n \geq 0$ *a.s. for* $n \geq 1$ *and* $\{X_n\} \uparrow \mathrm{cx}$, *or* $Y_n \geq 0$ *a.s. for* $n \geq 1$ *and* $\{Y_n\} \uparrow \mathrm{cx}$, *then* $\leq_{\mathcal{L}_3}=\leq_{\mathrm{cx}}$.

In the statements of the above propositions, we assumed the positivity (in mean or almost surely) of the sequences of $\{X_n\}$ or $\{Y_n\}$. Results can be established for negative random variables. In Tables 1 — 5 we provide, for easier reference, lists of statements equivalent to Propositions 3.2 — 3.6. These are easily proved by changing the signs of X_n and Y_n, then exchanging the roles of $\{X_n\}$ and $\{Y_n\}$ and/or using Lemma 2.3 to reverse the direction of the orderings.

	$\leq_{\mathcal{L}_1}$	$\leq_{\mathcal{L}_2}$	Conditions	$\leq_{\mathcal{L}_3}$
(a)	$\leq_{\mathrm{st}}$	$\leq_{\mathrm{st}}$	$Y_n \geq 0$ *a.s.*	$\leq_{\mathrm{st}}$
(b)	$\geq_{\mathrm{st}}$	$\leq_{\mathrm{st}}$	$Y_n \leq 0$ *a.s.*	$\leq_{\mathrm{st}}$
(c)	$\leq_{\mathrm{st}}$	$\geq_{\mathrm{st}}$	$X_n \leq 0$ *a.s.*	$\geq_{\mathrm{st}}$
(d)	$\geq_{\mathrm{st}}$	$\geq_{\mathrm{st}}$	$X_n \geq 0$ *a.s.*	$\geq_{\mathrm{st}}$

Table 1: Statements equivalent to Proposition 3.2

	$\leq_{\mathcal{L}_1}$	$\leq_{\mathcal{L}_2}$	Conditions	$\leq_{\mathcal{L}_3}$
(a)	$\leq_{\mathrm{st}}$	$\leq_{\mathrm{icx}}$	$EX_n \geq 0$ or $EY_n \geq 0$	$\leq_{\mathrm{icx}}$
(b)	$\geq_{\mathrm{st}}$	$\leq_{\mathrm{icv}}$	$EX_n \leq 0$ or $EY_n \leq 0$	$\leq_{\mathrm{icv}}$
(c)	$\leq_{\mathrm{st}}$	$\geq_{\mathrm{icx}}$	$EX_n \leq 0$ or $EY_n \leq 0$	$\geq_{\mathrm{icv}}$
(d)	$\geq_{\mathrm{st}}$	$\geq_{\mathrm{icx}}$	$EX_n \geq 0$ or $EY_n \geq 0$	$\geq_{\mathrm{icx}}$

Table 2: Statements equivalent to Proposition 3.3

	$\leq_{\mathcal{L}_1}$	$\leq_{\mathcal{L}_2}$	Conditions	$\leq_{\mathcal{L}_3}$
(a)	$\leq_{\mathrm{icx}}$	$\leq_{\mathrm{icx}}$	either $X_n \geq 0$ *a.s.* and $\{X_n\} \uparrow$ icx or $Y_n \geq 0$ *a.s.* and $\{Y_n\} \uparrow$ icx	$\leq_{\mathrm{icx}}$
(b)	$\geq_{\mathrm{icx}}$	$\leq_{\mathrm{icv}}$	either $X_n \leq 0$ *a.s.* and $\{X_n\} \downarrow$ icv or $Y_n \leq 0$ *a.s.* and $\{Y_n\} \downarrow$ icv	$\leq_{\mathrm{icv}}$
(c)	$\leq_{\mathrm{icx}}$	$\geq_{\mathrm{icv}}$	either $X_n \leq 0$ *a.s.* and $\{X_n\} \downarrow$ icv or $Y_n \leq 0$ *a.s.* and $\{Y_n\} \downarrow$ icv	$\geq_{\mathrm{icv}}$
(d)	$\geq_{\mathrm{icx}}$	$\geq_{\mathrm{icx}}$	either $X_n \geq 0$ *a.s.* and $\{X_n\} \uparrow$ icx or $Y_n \geq 0$ *a.s.* and $\{Y_n\} \uparrow$ icx	$\geq_{\mathrm{icx}}$

Table 3: Statements equivalent to Proposition 3.4

	$\leq_{\mathcal{L}_1}$	$\leq_{\mathcal{L}_2}$	Conditions	$\leq_{\mathcal{L}_3}$
(a)	$\leq_{\text{icv}}$	$\leq_{\text{icv}}$	either $X_n \geq 0$ *a.s.* and $\{X_n\} \downarrow$ icv or $Y_n \geq 0$ *a.s.* and $\{Y_n\} \downarrow$ icv	$\leq_{\text{icv}}$
(b)	$\geq_{\text{icv}}$	$\leq_{\text{icx}}$	either $X_n \leq 0$ *a.s.* and $\{X_n\} \uparrow$ icx or $Y_n \leq 0$ *a.s.* and $\{Y_n\} \uparrow$ icx	$\leq_{\text{icx}}$
(c)	$\leq_{\text{icv}}$	$\geq_{\text{icx}}$	either $X_n \leq 0$ *a.s.* and $\{X_n\} \uparrow$ icx or $Y_n \leq 0$ *a.s.* and $\{Y_n\} \uparrow$ icx	$\geq_{\text{icx}}$
(d)	$\geq_{\text{icv}}$	$\geq_{\text{icv}}$	either $X_n \geq 0$ *a.s.* and $\{X_n\} \downarrow$ icv or $Y_n \geq 0$ *a.s.* and $\{Y_n\} \downarrow$ icv	$\geq_{\text{icv}}$

Table 4: Statements equivalent to Proposition 3.5

	$\leq_{\mathcal{L}_1}$	$\leq_{\mathcal{L}_2}$	Conditions	$\leq_{\mathcal{L}_3}$
(a)	$\leq_{\text{cx}}$	$\leq_{\text{cx}}$	either $X_n \geq 0$ *a.s.* and $\{X_n\} \uparrow$ cx or $Y_n \geq 0$ *a.s.* and $\{Y_n\} \uparrow$ cx	$\leq_{\text{cx}}$
(b)	$\leq_{\text{cx}}$	$\leq_{\text{cx}}$	either $X_n \leq 0$ *a.s.* and $\{X_n\} \uparrow$ cx or $Y_n \leq 0$ *a.s.* and $\{Y_n\} \uparrow$ cx	$\leq_{\text{cx}}$
(c)	$\leq_{\text{cv}}$	$\leq_{\text{cv}}$	either $X_n \leq 0$ *a.s.* and $\{X_n\} \downarrow$ cv or $Y_n \leq 0$ *a.s.* and $\{Y_n\} \downarrow$ cv	$\leq_{\text{cv}}$
(d)	$\leq_{\text{cv}}$	$\leq_{\text{cv}}$	either $X_n \geq 0$ *a.s.* and $\{X_n\} \downarrow$ cv or $Y_n \geq 0$ *a.s.* and $\{Y_n\} \downarrow$ cv	$\leq_{\text{cv}}$

Table 5: Statements equivalent to Proposition 3.6

To conclude, we note that Propositions 3.2 – 3.6 can be generalized as follows:

Proposition 3.7 *Let $\leq_{\mathcal{L}_1}$, $\leq_{\mathcal{L}_2}$ and $\leq_{\mathcal{L}_3}$ be three integral stochastic orderings. The relation $\leq_{\mathcal{L}_3}=\leq_{\mathcal{L}_2}$ holds under any one of the five following conditions:*

1. *(a)* $\leq_{\mathcal{L}_1} \Rightarrow \leq_{\mathrm{st}} \Rightarrow \leq_{\mathcal{L}_2}$ *and*
   (b) $Y_n \geq 0$ *a.s. for* $n \geq 1$;
2. *(a)* $\leq_{\mathcal{L}_1} \Rightarrow \leq_{\mathrm{st}}$ *and* $\leq_{\mathcal{L}_2} \Leftarrow \leq_{\mathrm{icx}}$, *and*
   (b) either $EX_n \geq 0$ *or* $EY_n \geq 0$ *for* $n \geq 1$;
3. *(a)* $\leq_{\mathcal{L}_1} \Rightarrow \leq_{\mathrm{icx}} \Rightarrow \leq_{\mathcal{L}_2}$, *and*
   (b) either $X_n \geq 0$ *a.s. for* $n \geq 1$ *and* $\{X_n\} \uparrow \mathrm{icx}$, *or* $Y_n \geq 0$ *a.s. for* $n \geq 1$ *and* $\{Y_n\} \uparrow \mathrm{icx}$;
4. *(a)* $\leq_{\mathcal{L}_1} \Rightarrow \leq_{\mathrm{icv}} \Rightarrow \leq_{\mathcal{L}_2}$, *and*
   (b) either $X_n \geq 0$ *a.s. for* $n \geq 1$ *and* $\{X_n\} \downarrow \mathrm{icv}$, *or* $Y_n \geq 0$ *a.s. for* $n \geq 1$ *and* $\{Y_n\} \downarrow \mathrm{icv}$;
5. *(a)* $\leq_{\mathcal{L}_1} \Rightarrow \leq_{\mathrm{cx}} \Rightarrow \leq_{\mathcal{L}_2}$, *and*
   (b) either $X_n \geq 0$ *a.s. for* $n \geq 1$ *and* $\{X_n\} \uparrow \mathrm{cx}$, *or* $Y_n \geq 0$ *a.s. for* $n \geq 1$ *and* $\{Y_n\} \uparrow \mathrm{cx}$;

As an application of the above proposition, we obtain the moment ordering on the partial sums of positive RV's $X_n, Y_n \in \mathbb{R}^+$:

$$\sum_1^A X_n \leq_{\mathrm{m}} \sum_1^B Y_n,$$

if $A \leq_{\mathrm{icx}} B$, $X_n \leq_{\mathrm{m}} Y_n$, $n \geq 1$, and if $\{X_n\}$ or $\{Y_n\}$ is a sequence of i.i.d. random variables.

Remark In the case $\leq_{\mathcal{L}_1}=\leq_{\mathcal{L}_2}=\leq_{\mathcal{L}_3}=\leq_{\mathrm{st}}$, the sequences $\{X_n\}$ and $\{Y_n\}$ can be made of dependent random variables, provided that $\leq_{\mathcal{L}_2}$ holds between the infinite sequences, instead of each element. The proof relies on Strassen's theorem, as in the proof of Proposition 4.3.

3.3 Proofs

We prove Propositions 3.2–3.6. The proof of Proposition 3.7 follows trivially.

Proof of Proposition 3.2. By Strassen's Theorem (cf. Proposition 2.7), there exist two random variables A', B' such that

$$A' =_d A, \quad B' =_d B \quad \text{and} \quad A' \leq B' \ \ a.s.,$$

Let f be increasing. Then

$$Ef(U) = Ef(\sum_1^{A'} X_n) \leq Ef(\sum_1^{A'} Y_n) \leq Ef(\sum_1^{B'} Y_n) = Ef(V),$$

where we used the convolution property for the first inequality, and the positivity of Y_n for the last inequality.

□

The proofs of the other propositions are treated together.

There are two possible schemes of the proofs. Both take an arbitrary function $f \in C_{\mathcal{L}3}$. Let $U_k = \sum_{i=1}^k X_i$ and $V_k = \sum_{i=1}^k Y_i$.

$$Ef(U) = \sum_{k=0}^{\infty} Ef(U_k)P(A = k) = Eg(A) ,$$

where $g(k) = Ef(U_k)$. Likewise, with $h(k) = Ef(V_k)$,

$$Ef(V) = \sum_{k=1}^{\infty} h(k)P(B = k) = Eh(B) .$$

The goal is to prove $U \leq_{\mathcal{L}_3} V$, i.e. $Ef(U) \leq Ef(V)$. We proceed as follows:

Scheme 1:

i/ Prove that the function $k \mapsto g(k)$ is in $C_{\mathcal{L}1}$. Then we have $\sum g(k)P(A = k) \leq \sum g(k)P(B = k)$;

ii/ Prove that $g(k) \leq h(k)$ for all k. This yields the result.

Scheme 2:

i/ Prove that $g(k) \leq h(k)$ for all k. This gives: $\sum g(k)P(A = k) \leq \sum h(k)P(A = k)$;

ii/ Prove that the function $k \mapsto h(k)$ is in $C_{\mathcal{L}1}$, hence the result.

In both schemes, we need to prove $g(k) \leq h(k)$, which is true when $f \in C_{\mathcal{L}2}$ (Lemma 3.8). We must therefore have $C_{\mathcal{L}2} \subseteq C_{\mathcal{L}3}$, which implies that the ordering $\leq_{\mathcal{L}_3}$ has to be weaker than or equal to $\leq_{\mathcal{L}_2}$.

The symmetry of both schemes explains the fact that Propositions 3.3–3.6 need assumptions on $\{X_n\}$ *or* on $\{Y_n\}$, and not on both. All the proofs will be given with the assumptions taken on $\{X_n\}$ and using Scheme 1.

Lemma 3.8 *Take $f \in C_{\mathcal{L}2}$. Then $g(k) \leq h(k)$ for all $k \geq 0$.*

Proof This is in fact the convolution property, using the independence assumption of the RV's in each of the sequences $\{X_n\}$ and $\{Y_n\}$.

$\square$

Lemma 3.9 (increasingness) *If either*

i) $EX_n \geq 0$ *for all* $n \geq 1$ *and* f *is increasing and convex, or*

ii) $X_n \geq 0$ *a.s. for all* $n \geq 1$ *and* f *is increasing,*

then g *is increasing.*

Proof With the first assumption: $EX_n \geq 0 \Rightarrow 0 \leq_{\text{icx}} X_n$. f is increasing and convex, therefore:

$$g(k+1) - g(k) = Ef(U_k + X_{k+1}) - Ef(U_k) \geq Ef(U_k) - Ef(U_k) = 0.$$

Thus, g is increasing. With the second assumption, the inequality is obvious.

$\square$

Lemma 3.10 (convexity) *Assume* $X_n \geq 0, a.s.\ \forall n \geq 1$. *If either*

i) f *is convex and* $\{X_n\} \uparrow$ cx, *or*

ii) f *is increasing and convex, and* $\{X_n\} \uparrow$ icx,

then g *is convex.*

Proof Let f be a convex function. We have, for some RV's $Z_n =_d X_n$:

$$\begin{aligned}
g(k+1) - g(k) &= Ef(X_{k+1} + \sum_1^k X_n) - Ef(Z_k + \sum_1^{k-1} Z_n) \\
&\geq Ef(Z_k + \sum_1^k X_n) - Ef(Z_k + \sum_1^{k-1} Z_n) \\
&\geq Ef(\sum_1^k X_n) - Ef(\sum_1^{k-1} Z_n) \\
&= g(k) - g(k-1)
\end{aligned}$$

The first inequality results from the assumption that the sequence $\{X_n\}$ is increasing for the $\leq_{\text{cx}}$ order. Therefore, we have $Z_k \leq_{\text{cx}} X_{k+1}$. The second inequality simply results from the convexity of f, using $Z_n \geq 0$. The function g is convex. The proof is similar for the case where f is increasing convex and X_n increasing in the sense of the $\leq_{\text{icx}}$ ordering.

$\square$

Lemma 3.11 (concavity) *Assume* $X_n \geq 0, a.s.\ \forall n \geq 1$. *If either*

i) *f is concave and $\{X_n\} \downarrow$ cv, or*

ii) *f is increasing and concave, and $\{X_n\} \downarrow$ icv,*

then g is concave.

Proof The proof is as above: let f be a concave function:

$$\begin{aligned} g(k+1) - g(k) &= Ef(X_{k+1} + \sum_1^k X_n) - Ef(Z_k + \sum_1^{k-1} X_n) \\ &\leq Ef(Z_k + \sum_1^k X_n) - Ef(Z_k + \sum_1^{k-1} X_n) \qquad \text{(decreasingness of } \{X_n\} \text{)} \\ &\leq Ef(\sum_1^k X_n) - Ef(\sum_1^{k-1} Z_n) \qquad \text{(concavity of } f\text{)} \\ &= g(k) - g(k-1)\ . \end{aligned}$$

Thus g is concave.

□

Proof of Proposition 3.3. Take f increasing. Lemma 3.8 applies. As $EX_n \geq 0$, Lemma 3.9 applies also, so that $g \in \mathcal{C}_{\mathcal{L}1}$.

□

Proof of Proposition 3.4. Take f increasing and convex. As $\mathcal{C}_{\mathcal{L}3} \subseteq \mathcal{C}_{\mathcal{L}2}$ anyway, Lemma 3.8 applies. If $X_n \geq 0$ and $\{X_n\} \uparrow$ icx, Lemmas 3.9 and 3.10 apply also, and g is increasing and convex.

□

Proof of Proposition 3.5. Take f increasing and concave. As $\mathcal{C}_{\mathcal{L}3} \subseteq \mathcal{C}_{\mathcal{L}2}$, Lemma 3.8 applies. If $X_n \geq 0$ and $\{X_n\} \downarrow$ icv, Lemmas 3.9 and 3.11 apply also, and g is increasing and concave.

□

Proof of Proposition 3.6. Take f convex. As $\leq_{\mathcal{L}_1} = \leq_{\text{cx}}$, Lemma 3.8 applies. If $X_n \geq 0$ and $\{X_n\} \uparrow$ cx, Lemma 3.10 applies as well, and g is convex, i.e. $g \in \mathcal{C}_{\mathcal{L}1}$.

This Proposition can also be shown directly from Proposition 3.4 and the facts that $EU = EA \cdot EX_1$, $EV = EB \cdot EY_1$ (a type of Wald's identity) and that $EA = EB$, $EX_1 = EY_1$, which entail $EU = EV$.

□

4 Vectors of Random Partial Sums

In this section, we study the relationship between the stochastic orderings on the random variables and those on the vectors of the random partial sums of these random variables.

Let

- $\{a_n\}_1^\infty$ and $\{b_n\}_1^\infty$ be two sequences of positive integer valued RV's, mutually independent in each of the sequences,
- $\{X_n\}_1^\infty$ and $\{Y_n\}_1^\infty$ be two sequences of real valued RV's, mutually independent in each of the sequences,
- $A_n = \sum_1^n a_i$ and $B_n = \sum_1^n b_i$,
- $S_n = \sum_1^{A_n} X_i$ and $T_n = \sum_1^{B_n} Y_i$,
- $U = (S_1, \cdots, S_N)$, $V = (T_1, \cdots, T_N)$, where N is an arbitrary positive integer.

As in the previous section, all random variables used in this section are defined on a common probability space.

Assume that

- $\leq_{\mathcal{L}_1}, \leq_{\mathcal{L}_2}$ are stochastic orderings generated by the classes which are stable with respect to translations,
- $a_n \leq_{\mathcal{L}_1} b_n$ for all $n \geq 1$,
- $X_n \leq_{\mathcal{L}_2} Y_n$ for all $n \geq 1$,
- the sets of RV's $\{\{a_n\}_1^\infty, \{b_n\}_1^\infty\}$ and $\{\{X_n\}_1^\infty, \{Y_n\}_1^\infty\}$ are mutually independent.

We are interested in finding a stochastic ordering $\leq_{\mathcal{L}_3}$ such that $U \leq_{\mathcal{L}_3} V$.

We first prove the following general result:

Proposition 4.1 *Let $\{C_{\mathcal{R}^n}\}_{n=1}^\infty$ be a sequence of classes of functions such that*

(a) for all $x \in \mathbb{R}$, the mapping $(x_1, \ldots, x_{n-1}) \mapsto f(x_1, \ldots, x_{n-1}, x_{n-1} + x)$ belongs to $C_{\mathcal{R}^{n-1}}$,

(b) for all $(x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$, the mapping $x \mapsto f(x_1, \ldots, x_{n-1}, x_{n-1} + x)$ belongs to $C_{\mathcal{R}^1}$.

The ordering associated with $C_{\mathcal{R}^n}$ is noted $\leq_{\mathcal{R}^n}$. Assume that there is an ordering $\leq_{\mathcal{L}} \in \{\leq_{st}, \leq_{icx}, \leq_{icv}, \leq_{cx}\}$ such that: $\leq_{\mathcal{L}_1} \Rightarrow \leq_{\mathcal{L}} \Rightarrow \leq_{\mathcal{L}_2}$. Assume further that either of the sequences $\{X_n\}_{n=1}^\infty$ or $\{Y_n\}_{n=1}^\infty$ is composed of positive i.i.d. RV's. If $\leq_{\mathcal{R}^1} = \leq_{\mathcal{L}_2}$. then for all n,

$$(S_1, \ldots, S_n) \leq_{\mathcal{R}^n} (T_1, \ldots, T_n). \tag{1}$$

Proof We show the assertion by induction on n. Without loss of generality, we assume that the sequence $\{X_n\}$ is composed of positive i.i.d. RV's.

For $n = 1$, the relation $S_1 \leq_{\mathcal{R}^1} T_1$ follows immediately from Proposition 3.7. Suppose that for some $n \geq 1$, relation (1) holds. Let

$$S = (S_1, \cdots, S_n), \quad T = (T_1, \cdots, T_n), \quad s = (s_1, \cdots, s_n),$$

$$B = (B_1, \cdots, B_n), \quad \beta = (\beta_1, \cdots, \beta_n).$$

Then for all function $f \in C_{\mathcal{R}^{n+1}}$,

$$\begin{aligned}
& E\left[f(T, T_{n+1})\right] \\
&= \sum_{\beta} P(B = \beta) \int_{\mathbb{R}^n} E\left[f(T, T_{n+1}) | T = s, B = \beta\right] dP(T = s | B = \beta) \\
&= \sum_{\beta} P(B = \beta) \int_{\mathbb{R}^n} E\left[f\left(T, s_n + \sum_{j=\beta_n+1}^{\beta_n + b_{n+1}} Y_j\right)\right] dP(T = s | B = \beta) \\
&\geq \sum_{\beta} P(B = \beta) \int_{\mathbb{R}^n} E\left[f\left(T, s_n + \sum_{j=\beta_n+1}^{\beta_n + a_{n+1}} X_j\right)\right] dP(T = s | B = \beta)
\end{aligned}$$

(according to condition (b) and Proposition 3.7)

$$= \sum_{\beta} P(B = \beta) \int_{\mathbb{R}^n} E\left[f\left(T, s_n + \sum_{j=1}^{a_{n+1}} X_j\right)\right] dP(T = s | B = \beta)$$

(using the i.i.d. assumption on $\{X_n\}$)

$$\begin{aligned}
&= \int_{\mathbb{R}^n} E\left[f\left(T, s_n + \sum_{j=1}^{a_{n+1}} X_j\right)\right] dP(T = s) \\
&\geq \int_{\mathbb{R}^n} E\left[f\left(S, s_n + \sum_{j=1}^{a_{n+1}} X_j\right)\right] dP(S = s)
\end{aligned}$$

(due to condition (b) and the induction assumption)

$$= E\left[f(S, S_{n+1})\right].$$

Therefore, relation (1) holds for all $n \geq 1$.

□

Remark: The above proposition still holds when the conditions on the sequence $\{C_{\mathcal{R}^n}\}_{n=1}^{\infty}$ are replaced by

(a) for all $x \in \mathbb{R}$, the mapping $(x_1, \ldots, x_{n-1}) \mapsto f(x, x + x_1, \ldots, x + x_{n-1})$ belongs to $C_{\mathcal{R}^{n-1}}$,

(b) for all $(x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$, the mapping $x \mapsto f(x, x + x_1, \ldots, x + x_{n-1})$ belongs to $C_{\mathcal{R}^1}$, The proof is similar: instead of conditioning on $\boldsymbol{B}$ and $\boldsymbol{T}$, we condition on b_1 and T_1 and use the induction assumption

$$(S_2, \cdots, S_{n+1})_{\{S_1=s_1\}} \leq_{\mathcal{R}^n} (T_2, \cdots, T_{n+1})_{\{b_1=\beta_1, T_1=s_1\}}.$$

Corollary 4.2 *Let* $\leq_{\mathcal{L}} \in \{\leq_{st}, \leq_{cx}, \leq_{cv}, \leq_{icx}, \leq_{icv}\}$. *Assume that* $\leq_{\mathcal{L}_1} = \leq_{\mathcal{L}}$, $\leq_{\mathcal{L}_2} = \leq_{\mathcal{L}}$. *If either of the sequences* $\{X_n\}_{n=1}^{\infty}$ *or* $\{Y_n\}_{n=1}^{\infty}$ *is composed of positive i.i.d. RV's, then* $\leq_{\mathcal{L}_3} = \leq_{\mathcal{L}}$.

When $\leq_{\mathcal{L}_1} = \leq_{st}$, the i.i.d. assumption is not necessary. We have the following stronger results:

Proposition 4.3 *Assume that* $\leq_{\mathcal{L}_1} = \leq_{\mathcal{L}_2} = \leq_{st}$. *If* $Y_n \geq 0$ *a.s. for all* $n \geq 1$, *then* $\leq_{\mathcal{L}_3} = \leq_{st}$.

Proof According to Strassen's Theorem (Proposition 2.7), there exist sequences of random variables $\{a'_n\}_1^{\infty}$, $\{b'_n\}_1^{\infty}$, $\{X'_n\}_1^{\infty}$, and $\{Y'_n\}_1^{\infty}$ such that

$$\forall n \geq 1: \qquad a'_n =_d a_n, \quad b'_n =_d b_n, \quad X'_n =_d X_n, \quad Y'_n =_d Y_n,$$

and that

$$\forall n \geq 1: \qquad a'_n \leq b'_n \ \ a.s., \qquad X'_n \leq Y'_n \ \ a.s..$$

Using now the fact that $Y_n \geq 0$ *a.s.* (which implies that $Y'_n \geq 0$ *a.s.*) for all $n \geq 1$ yields that

$$\forall n \geq 1: \qquad A'_n = \sum_{i=1}^{n} a'_n \leq \sum_{i=1}^{n} b'_n = B'_n \ \ a.s.,$$

and that

$$\forall n \geq 1: \qquad S'_n = \sum_{i=1}^{A'_n} X'_n \leq \sum_{i=1}^{B'_n} Y'_n = T'_n \ \ a.s..$$

Therefore

$$U' = (S'_1, \cdots, S'_N) \leq (T'_1, \cdots, T'_N) = V' \ \ a.s..$$

Since in each of the sequences $\{a_n\}_1^{\infty}$, $\{b_n\}_1^{\infty}$, $\{X_n\}_1^{\infty}$, and $\{Y_n\}_1^{\infty}$, the random variables are mutually independent, we have that $\boldsymbol{U}' =_d \boldsymbol{U}$ and $\boldsymbol{V}' =_d \boldsymbol{V}$. Appealing again to Strassen's Theorem yields immediately

$$\boldsymbol{U} \leq_{st} \boldsymbol{V}.$$

□

Proposition 4.4 *Assume that* $\leq_{\mathcal{L}_1} = \leq_{st}$, $\leq_{\mathcal{L}_2} = \leq_{icx}$. *If* $Y_n \geq 0$ *a.s. for all* $n \geq 1$, *then* $\leq_{\mathcal{L}_3} = \leq_{icx}$.

Proof The proof is similar to the above one. According to Strassen's Theorem (Propositions 2.7 and 2.8), there exist sequences of random variables $\{a'_n\}_1^\infty$, $\{b'_n\}_1^\infty$, $\{X'_n\}_1^\infty$, and $\{Y'_n\}_1^\infty$ such that

$$\forall n \geq 1: \qquad a'_n =_d a_n, \quad b'_n =_d b_n, \quad X'_n =_d X_n, \quad Y'_n =_d Y_n,$$

and that

$$\forall n \geq 1: \qquad a'_n \leq b'_n \;\; a.s., \qquad X'_n \leq E[Y'_n | X'_n] \;\; a.s..$$

Using now the fact that $Y_n \geq 0$ $a.s.$ for all $n \geq 1$ yields that

$$\forall n \geq 1: \qquad A'_n = \sum_{i=1}^{n} a'_n \leq \sum_{i=1}^{n} b'_n = B'_n \;\; a.s.,$$

and that

$$\forall n \geq 1: \qquad S'_n = \sum_{j=1}^{A'_n} X'_j \leq E\left[\sum_{j=1}^{B'_n} Y'_j \mid \{X'_n\}\right] = E[T'_n \mid \{X'_n\}] \;\; a.s..$$

Therefore

$$U' = (S'_1, \cdots, S'_N) \leq E[(T'_1, \cdots, T'_N) \mid \{X'_n\}] = E[V' \mid \{X'_n\}] \;\; a.s..$$

According to Jensen's inequality on conditional expectation, we obtain that for all convex function $f : \mathbb{R}^{+^N} \to \mathbb{R}$,

$$E[f(U')] \leq E\left[f\left(E[V' \mid \{X'_n\}]\right)\right] \leq E\left[E\left[f(V') \mid \{X'_n\}\right]\right] = E[f(V')].$$

Using now the facts that $U' =_d U$ and that $V' =_d V$ yield

$$E[f(U)] = E[f(U')] \leq E[f(V')] = E[f(V)].$$

Therefore $U \leq_{\text{icx}} V$.

□

Remark: As in the previous section, equivalent statements of the above three propositions can be formulated for negative random variables.

5 Counting Measure of Random Intervals

We now turn to the problem of counting the number of occurrences of a point process (number of "arrivals") in random intervals the length distributions of which are stochastically ordered. This is in some sense the "dual" problem of the problem of random sums.

Let:

- P be a point process, and N be its associated counting measure,
- X and Y be two positive random variables, independent of P, with no mass at 0,

- $N_X = N(]0, X])$ and $N_Y = N(]0, Y])$.

Let $\leq_{\mathcal{L}} \in \{\leq_{st}, \leq_{cx}, \leq_{cv}, \leq_{icx}, \leq_{icv}\}$. Assume that $X \leq_{\mathcal{L}} Y$. What can we say of the ordering relations between N_X and N_Y ?

To answer this question, we will study several classes of point processes (stationary, renewal), both in continuous and discrete time, and try to obtain the strongest possible result in each case.

It will be shown below that for $\leq_{st}$ ordering, the point process can be of any kind. However, for the convex ordering ($\leq_{cx}, \leq_{icx}$), the ordering between N_X and N_Y depends strongly upon the distribution of the point process. The Poisson process has the particular property to be the only one among pure renewal processes to preserve the convex ordering.

5.1 General Processes

Without further assumptions on the process, we have the property:

Proposition 5.1 *If $X \leq_{st} Y$, then $N_X \leq_{st} N_Y$.*

Proof Using Strassen's Theorem, we have $X \leq Y a.s.$ on some probability space. As P is independent of X and Y, we can construct it on that same probability space. For each realization on this space, we will have $N_X(\omega) \leq N_Y(\omega)$, hence the result.

□

5.2 Stationary Renewal Processes

In this subsection, P will be a *stationary renewal* process with interarrivals τ following a distribution F: $F(x) = P(\tau \leq x)$. To avoid excessive complications, it is assumed that τ has a finite mean $E\tau$ and that F has no mass at 0 (i.e. no simultaneous arrivals). The first arrival occurs at τ_0 which follows the distribution $G(.)$ with

$$G(x) = \frac{1}{E\tau} \int_0^x (1 - F(t))dt.$$

It is well-known that for stationary processes, $E(N_t) = t/E\tau$. Therefore, by conditioning, we have $EX = EY \Rightarrow EN_X = EN_Y$. This allows the hope for a result with the convex ordering, and indeed:

Proposition 5.2 *Let $\leq_{\mathcal{L}} \in \{\leq_{cx}, \leq_{cv}, \leq_{icx}, \leq_{icv}\}$. If $X \leq_{\mathcal{L}} Y$, then $N_X \leq_{\mathcal{L}} N_Y$.*

Proof Let Φ_k denote the k-th fold convolution of F with itself. By convention, $\Phi_0 \equiv 1$. The probability that there are exactly k arrivals of the process in the interval $]0, t]$ is $G \star \Phi_{k-1}(t) - G \star \Phi_k(t)$ if $k > 0$ and $1 - G(t)$ if $k = 0$.

Take a function $f : \mathbb{N} \to \mathbb{R}$. We have, denoting by $X(.)$ the distribution function of X:

$$Ef(N_X) = \int_0^\infty g(x)X(dx), \qquad Ef(N_Y) = \int_0^\infty g(x)Y(dx),$$

where:

$$\begin{aligned} g(x) &= f(0) + \sum_{k=1}^{\infty} f(k)\,(G \star \Phi_{k-1}(x) - G \star \Phi_k(x)) \\ &= f(0) + \sum_{k=0}^{\infty} (f(k+1) - f(k))\, G \star \Phi_k(x)\ . \end{aligned}$$

It is clear that if f is increasing, so is g. Now observe that:

$$G \star \Phi_k(x) \;=\; \frac{1}{E\tau} \int_0^x (\Phi_k(t) - \Phi_{k+1}(t))\, dt\ .$$

This is easily seen from the definition of G. Replacing in $g(x)$, we have:

$$\begin{aligned} g(x) &= f(0) + \sum_{k=0}^{\infty} (f(k+1) - f(k)) \int_0^x (\Phi_k(t) - \Phi_{k+1}(t)) \frac{dt}{E\tau} \\ &= f(0) + (f(1) - f(0))\frac{x}{\tau} + \sum_{k=1}^{\infty} (f(k+1) - 2f(k) + f(k-1)) \int_0^x \Phi_k(t) \frac{dt}{E\tau} \end{aligned}$$

If f is convex, the coefficients of the series are positive. The integrals are increasing and convex functions of x because $\Phi_k(t)$ is increasing in t. This proves that g is convex, hence the result for $\leq_{\text{cx}}$ and $\leq_{\text{icx}}$. If f is concave, so is g, hence the result for $\leq_{\text{cv}}$ and $\leq_{\text{icv}}$.

□

Proposition 5.2 actually holds for any stationary process [4]. It has a counterpart for discrete time processes.

Let τ be an integer random variable such that $P(\tau = 0) = 0$. We define the "residual life" of τ as the variable τ_0 with distribution:

$$P(\tau_0 = k) = \frac{1}{E\tau} P(\tau \geq k), \qquad k \geq 1\ . \tag{2}$$

We call "stationary discrete renewal process" a process with independent interarrivals distributed according to some RV τ, and with the first arrival time distributed according to τ_0. Note that with this definition,

$$E\tau_0 = \frac{E\tau^2}{2E\tau} + \frac{1}{2},$$

which slightly differs from the continuous case.

Proposition 5.3 *Let P be a stationary discrete renewal process. Assume that X and Y are integer valued RV's. Let $\leq_{\mathcal{L}} \in \{\leq_{\text{cx}}, \leq_{\text{cv}}, \leq_{\text{icx}}, \leq_{\text{icv}}\}$. If $X \leq_{\mathcal{L}} Y$, then $N_X \leq_{\mathcal{L}} N_Y$.*

Proof The proof is similar to that of the time-continuous case. Let, for $k > 0$, $m \geq 0$: $\phi_k(m) = P(\tau_1 + \ldots + \tau_k = m)$, and $\phi_0(m) = \mathbf{1}_{\{m=0\}}$. Using the definition of τ_0, one obtains the probability of exactly k arrivals in $]0, n]$:

$$P(N_n = k) = \frac{1}{E\tau} \sum_{m=0}^{n} \sum_{l=0}^{m-1} (\phi_k(m) - \phi_{k+1}(m)), \qquad k \geq 0,\ n \geq 1\ .$$

Now:

$$Ef(N_X) = \sum_{n=0}^{\infty} g(n) P(X = n), \qquad \text{with :}\ \ g(n) = \sum_{k=0}^{\infty} f(k) P(N_n = k).$$

Using the expression of $P(N_n = k)$ above, one has:

$$g(n) = f(0) + (f(1) - f(0))\frac{n}{E\tau} + \frac{1}{E\tau}\sum_{k=1}^{\infty}(f(k+1) - 2f(k) + f(k-1)) \sum_{m=1}^{n} \sum_{l=0}^{m-1} \phi_k(l)\ .$$

One can then conclude as above.

□

5.3 Pure Renewal Processes

In this subsection, the process will be a renewal process with interarrivals τ following some distribution F. There is an arrival in 0. Again, we assume that F has no mass at 0 (i.e. no simultaneous arrivals).

It turns out that these processes do not have the nice order-preserving property of the stationary processes. This can be seen immediately from the following counterexample.

Take the process deterministic with $\tau \equiv 1$. Take $X \equiv 1$ so that $N_X \equiv 1$. If Y takes the values $(2/3, 8/3)$ with respective probabilities $(2/3, 1/3)$, then $EY = 4/3 > EX$ and $EN_Y = 2/3 < EN_X$. Therefore, not even the ordering of the average is conserved, and this rules out all possibility of convex or increasing-convex ordering between N_X and N_Y.

Expectations are actually reversed for a large class of arrival processes, as we shall see in Proposition 5.5 below.

On the other hand, the result of the previous section, applied to the Poisson process, shows that the ordering between X and Y is passed on to N_X and N_Y: indeed, the Poisson process can be seen at the same time as a pure renewal process and a stationary renewal process.

Our purpose is therefore to find out if other processes can preserve an ordering between X and Y on to N_X and N_Y.

5.3.1 General Interarrivals

As in the renewal case, take a function $f : \mathbb{N} \to \mathbb{R}$. We have now:

$$Ef(N_X) = \int_0^\infty g(x)X(dx),$$

with:

$$g(x) = f(0) + \sum_{k=1}^{\infty}(f(k) - f(k-1))\ \Phi_k(t)\ . \tag{3}$$

In particular, if $\forall k,\ f(k) = k$ then $g(x) = \sum_1^\infty \Phi_k(x)$ is (up to the constant 1) the renewal function of the process, which will be denoted by $g_0(x)$ from now on.

Let us first state a simple lemma showing the importance of the convexity of g:

Lemma 5.4 *Let f be a function $\mathbb{N} \to \mathbb{R}$. Let g be defined by (3): if the function g is strictly convex (resp. concave) on some interval $[a, b]$, then there exist RV's X and Y such that $X \leq_{\text{cx}} Y$ and $Ef(N_X) < Ef(N_Y)$ (resp. $Ef(N_X) > Ef(N_Y)$).*

Proof Assume that g is convex on $[a, b]$. Take any Y with support included in $[a, b]$, and $X = EY$. Of course, $X \leq_{\text{cx}} Y$. Now, using Jensen's inequality, $Ef(N_X) = g(EY) < Eg(Y) = Ef(N_Y)$. Likewise, if g is concave on $[a, b]$, then $Ef(N_X) > Ef(N_Y)$.
□

This lemma has an obvious counterpart in the discrete time case.

Results concerning the concavity of a renewal function are not abundant. The only result known to us is for the class of DFR distribution, and was first proved by Brown [9] (see also [13], [17]):

Proposition 5.5 (Brown) *If τ has a distribution with decreasing failure rate (DFR), then g_0 is concave.*

Corollary 5.6 *If τ has a DFR distribution, and if $X \leq_{\text{cx}} Y$, then $EN_X \geq EN_Y$.*

Proof According to Lemma 2.3, $X \leq_{\text{cx}} Y$ is equivalent to $Y \leq_{\text{cv}} X$. If τ has a DFR distribution, then g_0 is concave and $EN_X = Eg_0(X) \geq Eg_0(Y) = EN_Y$.
□

5.3.2 Exponential Interarrivals

Our first result states, quite in accordance with intuition, that the Poisson process alone preserves the $\leq_{\text{cx}}$ order.

Lemma 5.7 *If for any RV's X, Y such that $X \leq_{\rm cx} Y$ we have $EN_X = EN_Y$, then τ is exponentially distributed.*

Proof According to Lemma 5.4, the function g_0 can not be strictly convex or concave anywhere. As g_0 is left continuous and $g_0(0) = 0$, it has to be linear. The Poisson process is the only pure renewal process with a linear renewal function.

□

The above lemma and Lemma 2.3 imply that

Proposition 5.8 *The relation $N_X \leq_{\rm cx} N_Y$ holds for any RV's $X, Y \in \mathbb{R}$ such that $X \leq_{\rm cx} Y$ if and only if τ is exponentially distributed. The same holds for $\leq_{\rm cv}$.*

In the discrete time case, the same argument gives the corresponding result:

Proposition 5.9 *The relation $N_X \leq_{\rm cx} N_Y$ holds for any RV's $X, Y \in \mathbb{N}$ such that $X \leq_{\rm cx} Y$ if and only if τ is geometrically distributed on $\{1, 2 \ldots\}$. The same property holds for $\leq_{\rm cv}$.*

5.3.3 Interarrivals with Convex Renewal Functions

The following lemma and its corollary show that the convexity of the renewal function plays an essential role in the propagation of the $\leq_{\rm icx}$ order.

Lemma 5.10 *Assume that g_0 is convex on the interval $[0, \alpha]$. Then for all RV's X, Y with support in $[0, \alpha]$, the relation $X \leq_{\rm icx} Y$ implies $N_X \leq_{\rm icx} N_Y$.*

Proof Let a be a positive integer. Let $f(k) = [k - a]^+$ and g_a be the function defined by (3). If for all a, the function g_a is convex, the lemma is proved.

Clearly, $g_a(x) = \sum_{a+1}^{\infty} \Phi_k(x) = \Phi_a(x) \star g_0(x)$. As F has no mass in 0, $\Phi_a(0) = 0$ for any $a \geq 1$, and $g_0(0) = 0$. Consequently, we have:

$$g_a''(x) = \int_0^x g_0''(x - t)\ \Phi_a'(t)dt\ +\ g_0'(0)\Phi_a'(x)\ .$$

wherever this derivative exists. The functions Φ_a and g_0 are increasing. Therefore, if $g_0'' \geq 0$ on $[0, \alpha]$, g_a has the same property.

□

Lemmas 5.4 and 5.10 imply:

Corollary 5.11 *The process P preserves the $\leq_{\rm icx}$ ordering if and only if its renewal function is convex.*

No general sufficient conditions are known for g_0 to be convex. Simple analysis provides a number of necessary conditions.

Assume that g_0 is convex. It should be clear that F has to be continuous. Next, assuming that $E\tau$ is finite, the study of g_0 in the neighborhood of 0 and $+\infty$ give the following conditions:

- F has a derivative at 0, and $F'(0) \leq 1/E\tau$;
- If F has two derivatives at 0, then $F''(0) + F'(0)^2 \geq 0$;
- If $F(x) \sim Ax^\alpha$, $x \to 0$, then $\alpha > 1$ and $A \leq 1/E\tau$;
- If τ has a finite variance σ^2, then $\sigma/E\tau \leq 1$.

This last condition is a consequence of the asymptotic result (from [11, p. 366]): $g_0(t) - t/E\tau \to \sigma^2/2(E\tau)^2 - 1/2$, $t \to \infty$.

5.3.4 Gamma Interarrivals

In order to investigate in greater detail the behavior of g_0 for other processes than Poisson or deterministic, we turn to Gamma distributions, a family which is stable under convolution, with the hope of obtaining explicit results. If $\tau \sim f_{\nu,\alpha}$, we have:

$$\Phi_k(x) = \int_0^x \frac{(\alpha t)^{k\nu-1}}{\Gamma(k\nu)} e^{-\alpha t} \alpha dt , \qquad k \geq 1,$$

and

$$g_0(x) = \int_0^x \sum_{k=1}^{\infty} \frac{(\alpha t)^{k\nu-1}}{\Gamma(k\nu)} e^{-\alpha t} \alpha dt .$$

The study of g_0 by analytical means turns out to be quite complicated, and yields only partial answers which we complement by numerical studies. The case of Erlang distributions is however completely settled. Our results are summarized as follows:

Proposition 5.12 (Classification of Gamma Processes)

1. *If $\nu < 1$, g_0 is concave.*
2. *If $\nu = 1$ (Poisson process), g_0 is convex.*
3. *If $\nu \in]1, 2[$, extensive numerical experiments indicate that g_0 is convex.*
4. *If $\nu = 2$ (Erlang-2 interarrivals), g_0 is convex.*
5. *If ν is an integer greater or equal to 3, g_0 is neither concave nor convex.*
6. *If $\nu \geq 2$, g_0 is convex near 0.*

7. *If ν belongs to an interval of the form $]2k+1, 2k+2[$, $k \geq 1$, g_0 is neither concave nor convex.*

8. *If ν belongs to an interval of the form $]2k, 2k+1[$, $k \geq 1$, numerical evidence shows that g_0 is neither concave nor convex.*

Proof Point 1 follows from proposition 5.5, since Gamma distributions are DFR for $\nu < 1$ ([18, p. 280]). Point 2 has already been mentioned. Points 3 and 8 are illustrated by Figures 1 and 2.

For points 4 and 5, It can be shown (e.g. [11, p. 495]) that for all integer ν:

$$g_0''(x) = \frac{\alpha^2}{\nu} \sum_{k=1}^{\nu-1} \omega^k (\omega^k - 1) e^{(\omega^k - 1)\alpha x}, \qquad \omega = e^{2i\pi/\nu}.$$

For $\nu = 2$, this is simply $g_0''(x) = \alpha^2 e^{-2\alpha x}$ which is always positive. For $\nu \geq 3$, this function can be shown to oscillate. Consider for instance the behavior of $g_0''(x)\exp((1 - \cos(2\pi/\nu))\alpha x)$ when x goes to infinity.

For point 6, it is easy to see that when $x \to 0$, $g_0''(x) \sim \alpha^\nu x^{\nu-2}/\Gamma(\nu-1)$, which is always positive near 0 if $\nu \geq 2$.

For point 7, we remark that function g_0 can be expressed using modified Mittag-Leffler functions [10, pp. 210–]. The asymptotic analysis of these functions gives:

$$g_0''(x) = \alpha^2 e^{-\alpha x} \left(\frac{(\alpha x)^{-\nu-1}}{\Gamma(-\nu)} + o(x^{-\nu-1}) \right), \qquad x \to \infty.$$

The function Γ is negative on intervals $]-2k-1, -2k[$, $k \geq 0$. On the other hand, Point 6 states that $g_0''(x)$ is always positive near 0 if $\nu \geq 2$. Hence the result.

$\square$

5.3.5 Other Families of Interarrivals

To conclude this section, we checked whether some families of RV's have a convex renewal function or not. The following results can be obtained through routine calculations on g_0 or its Laplace transform.

- If τ is uniformly distributed on $[a, b]$, g_0 is not convex: one can check that it is concave at $x = b$.
- If τ is the sum of two exponential RV's, then g_0 is convex.
- If τ is the sum of three exponential RV's with parameters λ_1, λ_2 and λ_3, then g_0 is convex if and only if $\lambda_1 \notin](\sqrt{\lambda_2} - \sqrt{\lambda_3})^2, (\sqrt{\lambda_2} + \sqrt{\lambda_3})^2[$.
- If τ is the sum of two Erlang-2 RV's with parameters λ_1 and λ_2, then g_0 is convex if, and only if $\lambda_1 \notin](3 - \sqrt{8})\lambda_2, (3 + \sqrt{8})\lambda_2[$.

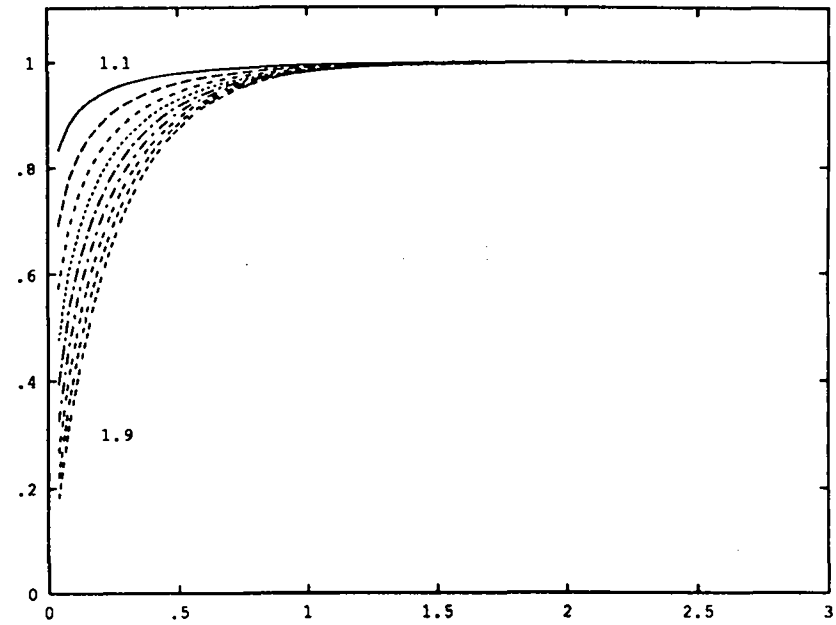

Figure 1: The function $g_0'(x)$ for ν from 1.1 to 1.9 with step 0.1

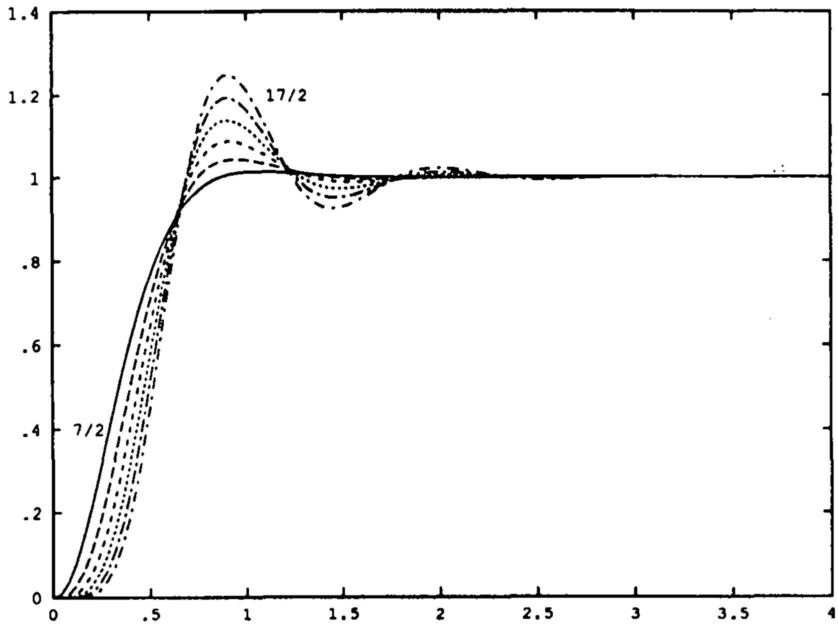

Figure 2: The function $g_0'(x)$ for ν from 7/2 to 17/2 with step 1

5.4 Preservation of the $\leq_{\text{icv}}$ ordering

According to Lemma 2.5, the preservation of the increasing and convex order is equivalent to: $(X \leq_{\text{icv}} Y \Rightarrow E[a - N_X]^+ \geq E[a - N_Y]^+, \ \forall a)$, or to $(X \leq_{\text{icv}} Y \Rightarrow E\max(a, N_X) \leq E\max(a, N_Y), \ \forall a \in \mathbb{N})$. Using the computations of the previous subsection, one obtains:

Proposition 5.13 *The process P preserves the $\leq_{\text{icv}}$ ordering if and only if for all $a \in \mathbb{N}$, the function $\sum_{k=1}^{a} \Phi_k(x)$ is concave.*

6 Applications

In this section, we apply the results established in the previous sections to various analysis and control problems arising in queueing systems. We obtain new and more general results for these application examples.

6.1 Busy Period of the $M/GI/1$

Let $\sigma^{(1)}$ and $\sigma^{(2)}$ be two random variables. Consider two $M/GI/1$ queues with the same arrival intensity, and for which the service time distribution is that of $\sigma^{(1)}$ and $\sigma^{(2)}$ respectively. Let $Y^{(i)}$ be a RV with the distribution of the busy period in steady state in queue i, $i = 1, 2$.

Proposition 6.1 *Let $\leq_{\mathcal{L}} \in \{\leq_{\text{st}}, \leq_{\text{cx}}, \leq_{\text{cv}}, \leq_{\text{icx}}, \leq_{\text{icv}}\}$. If $\sigma^{(1)} \leq_{\mathcal{L}} \sigma^{(2)}$, then $Y^{(1)} \leq_{\mathcal{L}} Y^{(2)}$.*

Proof Let ν_i be the number of arrivals of the Poisson process in an random interval whose length is distributed according to $\sigma^{(i)}$. The variables $Y^{(i)}, i = 1, 2$ are known to satisfy the equation ([12, p. 211]):

$$Y^{(i)} = \sigma^{(i)} + \sum_{j=1}^{\nu_i} Y_j^{(i)} , \tag{4}$$

where the $Y_j^{(i)}$ are i.i.d. random variables with the same distribution as $Y^{(i)}$. Let now, for i=1,2, $\{(\sigma_n^{(i)}, \nu_n^{(i)})\}_0^\infty$ be a sequence of i.i.d. RV's distributed according to $(\sigma^{(i)}, \nu^{(i)})$. Consider the scheme:

$$\begin{aligned} Z_0^{(i)} &= \sigma_0^{(i)} \\ Z_{n+1}^{(i)} &= \sigma_{n+1}^{(i)} + \sum_{j=1}^{\nu_{n+1}^{(i)}} Z_{n,j}^{(i)} , \qquad n \geq 0, \end{aligned}$$

where the $Z_{n,j}^{(i)}$ are i.i.d. RV's with the same distribution as $Z_n^{(i)}$. We claim:

Claim 1: $\forall n \geq 0, Z_n^{(i)} \leq_{st} Z_{n+1}^{(i)}$.

We have obviously $Z_0^{(i)} \leq_{st} Z_1^{(i)}$. By induction, if $Z_{n-1}^{(i)} \leq_{st} Z_n^{(i)}$, then because $(\sigma_n^{(i)}, \nu_n^{(i)}) =_d (\sigma_{n+1}^{(i)}, \nu_{n+1}^{(i)})$, and because the RV's $Z_{n-1,j}^{(i)}$ and $Z_{n,j}^{(i)}$ are assumed to be independent of the other variables, we can apply Strassen's theorem to obtain $Z_n^{(i)} \leq_{st} Z_{n+1}^{(i)}$.

Claim 2: $Z_n^{(i)}$ converges in law to a proper RV with the distribution of $Y^{(i)}$.

The sequence $Z_n^{(i)}$ is stochastically increasing, therefore it converges. That the limit is a proper RV stems from the fact that $EZ_{n+1}^{(i)} = E\sigma^{(i)} + \lambda E\sigma^{(i)} EZ_n^{(i)}$ and that this recurrence converges if $\lambda E\sigma^{(i)} < 1$. The limit satisfies equation (4) which is known to have only one solution.

Claim 3: If $\sigma^{(1)} \leq_{\mathcal{L}} \sigma^{(2)}$, then $\forall n \geq 0, Z_n^{(1)} \leq_{\mathcal{L}} Z_n^{(2)}$.

We prove this claim by induction on n. Clearly, $Z_0^{(1)} \leq_{\mathcal{L}} Z_0^{(2)}$. Assume that the claim holds for some $n \geq 0$. According to Propositions 5.1 and 5.2, $\nu^{(1)} \leq_{\mathcal{L}} \nu^{(2)}$, Using Propositions 3.2—3.6 implies that $\sum_{j=1}^{\nu_{n+1}^{(1)}} Z_{n,j}^{(1)} \leq_{\mathcal{L}} \sum_{j=1}^{\nu_{n+1}^{(2)}} Z_{n,j}^{(2)}$. However, this does not allow us to conclude that $Z_{n+1}^{(1)} \leq_{\mathcal{L}} Z_{n+1}^{(2)}$, as $\sigma^{(i)}$ and $\nu^{(i)}$ are correlated. Let $f \in C_{\mathcal{L}}$.

$$Ef(Z_{n+1}^{(1)}) = \int Ef(s + \sum_{j=1}^{N(s)} Z_{n,j}^{(1)})\, dP(\sigma^{(1)} = s) ,$$

where $N(s)$ is the (random) number of arrivals of the Poisson process in the (fixed) interval $]0, s]$. Clearly, $\sum_{j=1}^{N(s)} Z_{n,j}^{(1)} \leq_{\mathcal{L}} \sum_{j=1}^{N(s)} Z_{n,j}^{(2)}$ for all fixed s. As the function $z \mapsto f(s+z)$ is in $C_{\mathcal{L}}$, we obtain:

$$Ef(Z_{n+1}^{(1)}) \leq \int Ef(s + \sum_{j=1}^{N(s)} Z_{n,j}^{(2)})\, dP(\sigma^{(1)} = s) .$$

To conclude, we need to show that the mapping $g : s \mapsto Ef(s + \sum_{j=1}^{N(s)} Z_{n,j}^{(2)})$ is in $C_{\mathcal{L}}$. Let λ be the intensity of the Poisson process. We have:

$$g(s) = \sum_{k=0}^{\infty} f_k(s) \frac{(\lambda s)^k}{k!} e^{-\lambda s}, \quad \text{with} \quad f_k(s) = Ef(s + \sum_{1}^{k} Z_{n,j}^{(2)}) .$$

By differentiation,

$$\begin{aligned} g'(s) &= \sum_{k=0}^{\infty} [f_k'(s) + \lambda(f_{k+1}(s) - f_k(s))] \frac{(\lambda s)^k}{k!} e^{-\lambda s}, \\ g''(s) &= \sum_{k=0}^{\infty} [f_k''(s) + \lambda(f_{k+1}'(s) - f_k'(s)) + \lambda^2(f_{k+2}(s) - 2f_{k+1}(s) + f_k(s))] \\ &\qquad \frac{(\lambda s)^k}{k!} e^{-\lambda s}. \end{aligned}$$

If f is increasing, then (cf. Lemma 3.9) $f_k(s)$ is increasing in both s and k. Therefore, g is increasing. If f is convex, then $f_k(s)$ is convex in s and in k (cf. Lemma 3.10) and, because f' is increasing, $f'_k(s)$ is increasing in k. If f is concave, then (cf. Lemma 3.11) $f_k(s)$ is concave in both k and s, and as $-f'$ is increasing, $f'_k(s)$ is decreasing in k. Claim 3 is therefore proved.

According to Claims 2 and 3, the relation $Y^{(1)} \leq_{\mathcal{L}} Y^{(2)}$ holds if the $\leq_{\mathcal{L}}$ ordering is preserved for the limit RV's of weakly convergent sequences of RV's. In case $\leq_{\mathcal{L}}=\leq_{\mathrm{st}}$, this is trivially true. For the cases $\leq_{\mathcal{L}}=\leq_{\mathrm{cx}}, \leq_{\mathrm{icx}}$, one can verify that the conditions of Proposition 1.3.2. of Stoyan [19] are fulfilled. For the cases $\leq_{\mathcal{L}}=\leq_{\mathrm{cv}}, \leq_{\mathrm{icv}}$, one can still apply Proposition 1.3.2. of Stoyan [19] by changing the signs of the RV's (cf. Lemma 2.3).

□

The proposition easily extends to busy periods with an initial load. For i=1,2, let $\hat{Y}^{(i)}$ be the first busy period of an $M/GI/1$ system with services distributed according to $\sigma^{(i)}$. Assume that the queue has initial unfinished work $W^{(i)}(0)$, or an initial number of customers $N^{(i)}(0)$ *and* that it starts a service at $t = 0$.

Corollary 6.2 *Let* $\leq_{\mathcal{L}} \in \{\leq_{\mathrm{st}}, \leq_{\mathrm{cx}}, \leq_{\mathrm{cv}}, \leq_{\mathrm{icx}}, \leq_{\mathrm{icv}}\}$. *If* $\sigma^{(1)} \leq_{\mathcal{L}} \sigma^{(2)}$, *and either* $W^{(1)}(0) \leq_{\mathcal{L}} W^{(2)}(0)$ *or* $N^{(1)}(0) \leq_{\mathcal{L}} N^{(2)}(0)$, *then* $\hat{Y}^{(1)} \leq_{\mathcal{L}} \hat{Y}^{(2)}$.

Proof If $N^{(1)}(0) \leq_{\mathcal{L}} N^{(2)}(0)$ and $\sigma^{(1)} \leq_{\mathcal{L}} \sigma^{(2)}$, then from the results of Section 3, $W^{(1)}(0) \leq_{\mathcal{L}} W^{(2)}(0)$. The busy period is now given by the equation:

$$\hat{Y}^{(i)} = W_0^{(i)} + \sum_{j=1}^{\nu_i} Y_j^{(i)} ,$$

where ν_i is the number of arrivals in the interval $]0, W^{(i)}(0)]$ and the RV's $Y_j^{(i)}$ are independent and identically distributed according to a *standard* busy period. Given Proposition 6.1, the same proof as in Claim 3 above yields the result.

□

6.2 Queues with Non-Stationary Inputs

This example illustrates the interest of relaxing the assumptions of Proposition 3.1 to the monotonicity of only one of the sequences $\{X_n\}$ and $\{Y_n\}$.

Let X be an integer valued random variable. Consider a single server queue with batch arrivals (denoted by $G^X/G/1$), with *i.i.d.* bulk sizes distributed according to X, interarrival sequence $\{\tau_n^{(1)}\}_0^\infty$ and service time sequence $\{\sigma_n^{(1)}\}_0^\infty$. These two random sequences are mutually independent but are not identically distributed. Let $\{W_n^{(1)}\}_0^\infty$ be the sequence of successive bulk waiting times, with $W_0^{(1)} = 0$. The goal of this subsection is to find systems with stationary input that provide (closest) lower and upper bounds in the sense of $\leq_{\mathrm{icx}}$ for the waiting times.

These systems are constructed with the RV's $\tau^{(2)}, \sigma^{(2)}$ and $\tau^{(3)}, \sigma^{(3)}$ such that

- $\forall n \geq 0$, $-\tau^{(2)} \leq_{\rm icx} -\tau_n^{(1)} \leq_{\rm icx} -\tau^{(3)}$, and $\sigma^{(2)} \leq_{\rm icx} \sigma_n^{(1)} \leq_{\rm icx} \sigma^{(3)}$.
- For any RV τ and σ, if $\forall n \geq 0$, $-\tau \leq_{\rm icx} -\tau_n^{(1)}$, and $\sigma \leq_{\rm icx} \sigma_n^{(1)}$, then $-\tau \leq_{\rm icx} -\tau^{(2)}$, and $\sigma \leq_{\rm icx} \sigma^{(2)}$.
- For any RV τ and σ, if $\forall n \geq 0$, $-\tau_n^{(1)} \leq_{\rm icx} -\tau$, and $\sigma_n^{(1)} \leq_{\rm icx} \sigma$, then $-\tau^{(3)} \leq_{\rm icx} -\tau$, and $\sigma^{(3)} \leq_{\rm icx} \sigma$.

Let $\{W_n^{(2)}\}_0^\infty$ (resp. $\{W_n^{(3)}\}_0^\infty$) be the sequence of bulk waiting times in the $GI^X/GI/1$ queue with $W_0^{(2)} = 0$ (resp. $W_0^{(3)} = 0$) and with the interarrivals and the service times distributed according to $\tau^{(2)}$ and $\sigma^{(2)}$ (resp. $\tau^{(3)}$ and $\sigma^{(3)}$) respectively.

It then follows from Corollary 4.2 and Loynes' scheme that

$$\forall n \geq 0, \qquad (W_0^{(2)}, \cdots, W_n^{(2)}) \leq_{\rm icx} (W_0^{(1)}, \cdots, W_n^{(1)}) \leq_{\rm icx} (W_0^{(3)}, \cdots, W_n^{(3)}). \tag{5}$$

The existence of the RV's $\tau^{(2)}, \sigma^{(2)}$ and $\tau^{(3)}, \sigma^{(3)}$ are guaranteed by Proposition 6.3 below.

We use the following notation: let Y be a positive RV. Its complementary distribution function is $\overline{F}_Y(t) = P(Y > t)$ and we note $\phi_Y(x) = \int_x^\infty \overline{F}_Y(t)dt$. It is easy to see that for $x \geq 0$, ϕ_Y is a continuous, decreasing, convex function which tends to 0 when x goes to infinity. Conversely, if ϕ is a function that satisfies these properties, then there is a positive RV Z such that $\phi = \phi_Z$.

Proposition 6.3 *Let Y_1 and Y_2 be two positive random variables. Let Z_1 be the RV such that ϕ_{Z_1} is the convex hull of ϕ_X and ϕ_Y on $\mathbb{R}^+$, and Z_2 be the RV such that $\phi_{Z_2} = \max(\phi_X, \phi_Y)$ on $\mathbb{R}^+$. Then*

i) $Z_1 \leq_{\rm icx} Y_1 \leq_{\rm icx} Z_2$ *and* $Z_1 \leq_{\rm icx} Y_2 \leq_{\rm icx} Z_2$,

ii) *if $EY_1 = EY_2$, then $Z_1 \leq_{\rm cx} Y_1 \leq_{\rm cx} Z_2$ and $Z_1 \leq_{\rm cx} Y_2 \leq_{\rm cx} Z_2$,*

iii) *For all RV $Z \in \mathbb{R}^+$, if $Z \leq_{\rm icx} Y_1$ and $Z \leq_{\rm icx} Y_2$, then $Z \leq_{\rm icx} Z_1$,*

iv) *For all RV $Z \in \mathbb{R}^+$, if $Y_1 \leq_{\rm icx} Z$ and $Y_2 \leq_{\rm icx} Z$, then $Z_2 \leq_{\rm icx} Z$.*

Proof Both functions $\max(\phi_X, \phi_Y)$ and the convex hull of ϕ_X and ϕ_Y are continuous and convex functions, decreasing to 0, and Z_1 and Z_2 are well defined.

Claim i) follows from the well known characterization of the $\leq_{\rm icx}$ ordering: $A \leq_{\rm icx} B \Leftrightarrow \phi_A \leq \phi_B$. Claim ii) follows immediately from the fact that if EY exists, $\phi_Y(0) = EY$ so that if $EY_1 = EY_2$, then $EZ_1 = EZ_2 = EY_1$.

For Claims iii) and iv), we use Stoyan's result [19, p. 9] that if f is increasing and convex on $\mathbb{R}^+$ with $f(0) = 0$, then there exists an increasing function g such that $f(t) = \int_0^t g(u)du$. Thus, $Ef(X) = \int_0^\infty \phi_X(t)dg(t)$, provided the expectation exists. This property is easily extended to functions with $f(0) \neq 0$. The Claims iii) and iv) now follow immediately from the facts that $dg \geq 0$ and that ϕ_{Z_1} (resp. ϕ_{Z_2}) is the largest (resp. smallest) convex function smaller (resp. greater) than both ϕ_{Y_1} and ϕ_{Y_2}.

□

6.3 Comparison of Queues with Bulk Arrivals

In this subsection we study again queues with bulk arrivals. We obtain a lower bound (in the increasing convex ordering sense) for the stationary customer response time with a system with deterministic bulk size.

Let X be an integer valued random variable. Let $\overline{X} = \lfloor EX \rfloor$. Let $\{W_n^b\}_0^\infty$ be the sequence of *bulk* waiting times in a $GI^X/GI/1$ queue with i.i.d. bulk sizes distributed according to X. Similarly, let $\{\overline{W}_n^b\}_0^\infty$ be the sequence of bulk waiting times in the $GI^{\overline{X}}/GI/1$ queue with the same interarrival and service distributions, but with bulk size equal to $\overline{X}$. Let $\{\sigma_n\}_0^\infty$ be the sequence of service times of both queues. It is clear that the bulk waiting time in a $GI^X/GI/1$ queue are the same as customer waiting times in a $GI/GI/1$ queue with service times distributed according to the RV $\tilde{\sigma}_X = \sum_1^X \sigma_i$. Obviously, $\overline{X} \leq_{\text{icx}} X$, and by Corollary 4.2 we can prove, using Loynes' scheme, that

Proposition 6.4 (Ordering of vectors of bulk waiting times in transient regime) *For all $n \geq 0$:*

$$(\overline{W}_0^b, \ldots, \overline{W}_n^b) \leq_{\text{icx}} (W_0^b, \ldots, W_n^b) \, .$$

We now turn to customer response times in stationary regime, denoted by R^c and $\overline{R}^c$, corresponding to the systems with bulk sizes X and $\overline{X}$, respectively. If W^b and $\overline{W}^b$ are the steady state waiting times of bulks, we have:

$$R^c = W^b + \sum_{i=1}^{C(X)} \tau_i \, , \tag{6}$$

where $C(X)$, the "typical number" of a customer in a bulk, is distributed as: $P(C(X) = k) = P(X \geq k)/EX$. Taking expectations:

$$ER^c = EW^b + E\tau \left(\frac{1}{2} + \frac{\sigma_X^2 + (EX)^2}{2EX} \right) ,$$

whereas

$$E\overline{R}^c = E\overline{W}^b + E\tau \left(\frac{1}{2} + \frac{(EX)^2}{2EX} \right) .$$

Thus, $ER^c \geq E\overline{R}^c$. In fact, we propose a much stronger result:

Proposition 6.5 (Comparison of stationary customer response times)

$$\overline{R}^c \leq_{\text{icx}} R^c. \tag{7}$$

Proof Introduce the integer valued random variable $\hat{X}$ defined as:

$$\hat{X} = \begin{cases} \overline{X} & \text{with probability } p = \overline{X} + 1 - EX \\ \overline{X}+1 & \text{with probability } 1-p \end{cases} .$$

It is easy to see that $\overline{X} \leq_{st} \hat{X}$ (with equality if EX is integer). Denote by $\tilde{\sigma}_{\overline{X}} = \sum_1^{\overline{X}} \sigma_i$ and $\tilde{\sigma}_{\hat{X}} = \sum_1^{\hat{X}} \sigma_i$. Owing to Proposition 3.2, $\tilde{\sigma}_{\overline{X}} \leq_{st} \tilde{\sigma}_{\hat{X}}$, which implies, as in Proposition 6.4, that for all n,

$$(\overline{W}_0^b, \ldots, \overline{W}_n^b) \leq_{st} (\widehat{W}_0^b, \ldots, \widehat{W}_n^b).$$

It is easy to see that when $E\tau > EX \cdot E\sigma$ (the standard stability condition), $\overline{W}_n^b$ and $\hat{W}_n^b$ weakly converge to the stationary bulk waiting times $\overline{W}^b$ and $\hat{W}_b$, respectively, when n goes to infinity. Furthermore, the relation $\overline{W}_n^b \leq_{st} \hat{W}_n^b$, $\forall n \geq 1$ implies:

$$\overline{W}^b \leq_{st} \widehat{W}^b. \tag{8}$$

We have $E\hat{X} = EX$ and actually we prove that: $\hat{X} \leq_{cx} X$. Indeed, let $f : \mathbb{R} \to \mathbb{R}$, be a convex function and let $\overline{f} : \mathbb{R} \to \mathbb{R}$ be the piecewise linear function, interpolated from f at points $(n, f(n)), n \in \mathbb{N}$. Then $Ef(X) = E\overline{f}(X) \geq E\overline{f}(EX) = \overline{f}(EX)$, because $\overline{f}$ is convex and $EX \leq_{cx} X$. But $\overline{f}(EX) = pf(\overline{X}) + (1-p)f(\overline{X}+1) = Ef(\hat{X})$.

Consequently, from Proposition 3.6, $\tilde{\sigma}_{\hat{X}} \leq_{cx} \tilde{\sigma}_X$ and as in the above proposition, for all n,

$$(\widehat{W}_0^b, \ldots, \widehat{W}_n^b) \leq_{icx} (W_0^b, \ldots, W_n^b).$$

Owing to Proposition 1.3.2. of Stoyan [19], we obtain, for the stationary bulk waiting times, that

$$\widehat{W}^b \leq_{icx} W^b. \tag{9}$$

Relations (8) and (9) entail that

$$\overline{W}^b \leq_{icx} W^b. \tag{10}$$

In order to prove the $\leq_{icx}$ ordering between response times, we first show that

$$C(\hat{X}) \leq_{st} C(X). \tag{11}$$

This result is analogous to the implication $\leq_{cx} \Rightarrow \leq_{S-st}$ in continuous time (see. [5], [1]). The proof uses the characterization of the increasing convex ordering (see Section 6.2), and the definition of $C(X)$:

$$\begin{aligned} \hat{X} \leq_{icx} X \quad &\Leftrightarrow \quad \forall x, \int_x^\infty P(\hat{X} > t)dt \leq \int_x^\infty P(X > t)dt \\ &\Leftrightarrow \quad \forall n \in \mathbb{N}, \sum_{k=n}^{\infty} P(\hat{X} \geq k) \leq \sum_{k=n}^{\infty} P(\hat{X} \geq k) \end{aligned}$$

because both $\hat{X}$ and X belong to $\mathbb{N}$,

$$\Rightarrow \quad \forall n \in \mathbb{N}, \sum_{k=n}^{\infty} P(C(\widehat{X}) = k) \leq \sum_{k=n}^{\infty} P(C(X) = k)$$

$$\Leftrightarrow \quad \forall n \in \mathbb{N}, \; P(C(\widehat{X}) \geq n) \leq P(C(X) \geq n),$$

because $EX = E\widehat{X}$. Finally, this is equivalent to (11).

According to Lemma 6.6 below, $C(\overline{X}) \leq_{st} C(\widehat{X})$. This, together with (11) imply that $C(\overline{X}) \leq_{st} C(X)$.

Now, in view of (6) and Proposition 3.4, we finally obtain:

$$\overline{R}^c \leq_{icx} R^c .$$

□

Lemma 6.6 *Let X_1 and X_2 be real valued RV's with supports included in $[0, \alpha]$ and $[\alpha, \infty)$, respectively, for some $\alpha \geq 0$. Let C_1 and C_2 be the "residual lives" of X_1 and X_2, defined by:*

$$P(C_i \leq t) = \frac{1}{EX_i} \int_0^t P(X_i > u) du, \quad i = 1, 2.$$

Then:

$$C_1 \leq_{st} C_2 .$$

If X_1 and X_2 are integer valued, and the residual lives $C(X_1)$ and $C(X_2)$ are defined by (2), then:

$$C(X_1) \leq_{st} C(X_2) .$$

Proof The function $P(C_1 \leq t)$ is concave. As $P(C_1 \leq 0) \geq 0$, $P(C_1 \leq \alpha) = 1$, and by concavity, $P(C_1 \leq t) \geq t/\alpha$ for all $0 \leq t \leq \alpha$. Therefore, if $0 \leq t \leq \alpha$, since $EX_2 \geq \alpha$,

$$P(C_2 \leq t) = t/EX_2 \leq t/\alpha \leq P(C_1 \leq t).$$

If $t > \alpha$,

$$P(C_2 \leq t) \leq 1 = P(C_1 \leq t).$$

Hence the result. The proof for the discrete case is similar.

□

6.4 Comparison between Cyclic and Bernoulli Routing Polices in Transient Regime

Consider a system of K symmetric $./GI/1$ queues in parallel. Queues are numbered from 0 to $K-1$. The system is initially empty. Customers arrive according to a sequence of i.i.d. interarrivals $\{\tau_n\}_0^\infty$. Customer n requires a service σ_n, where $\{\sigma_n\}_0^\infty$ is a sequence of i.i.d. random variables. When a customer arrives to the system, it is immediately routed (allocated) to one of the queues.

We want to compare two routing policies:

- "Bernoulli", where customer number n joins queue N_n with $\{N_n\}_0^\infty$ being a sequence of i.i.d. RV's uniformly distributed on $\{0, 1, \ldots, K-1\}$.
- "Round Robin", where customer number n joins queue n mod K.

Let $\{W_n^B\}_0^\infty$ and $\{W_n^{RR}\}_0^\infty$ be the sequences of waiting times of the customers with Bernoulli and Round Robin routing, respectively.

Makowski and Philips [15] showed that under the natural stability condition, these sequences weakly converge as n goes to infinity, and that: $W_\infty^B \leq_{\text{icx}} W_\infty^{RR}$. We extend this result to the transient regime.

Proposition 6.7 (Comparison for the transient regime) *For all $n \geq 0$:*

$$W_n^{RR} \leq_{\text{icx}} W_n^B \,. \tag{12}$$

Proof The proof is based on the use of Loynes' scheme (cf. [2], [1] and [14]) to construct a $GI/GI/1$ system which is a lower bound for the system with Bernoulli routing.

For the Round Robin system, it is clear that the customer response times in each queue are the successive response times in the $GI/GI/1$ system with i.i.d. services $\{\sigma_n\}_0^\infty$ and i.i.d. interarrivals $\{\tau_n'\}_0^\infty$, where

$$\tau_n' =_d \sum_{i=1}^{K} \tau_i. \tag{13}$$

For the Bernoulli routing, define the K sequences $\{u_n^k\}_0^\infty$, $k = 0, \ldots, K-1$, by: $u_n^k = 1$ if $N_n = k$ and 0 otherwise. For each queue, define a "virtual waiting time" for customer n by the recurrence:

$$\begin{aligned} V_0^k &= 0, \\ V_{n+1}^k &= [V_n^k + u_n^k \sigma_n - \tau_n]^+, \qquad n \geq 0. \end{aligned} \tag{14}$$

The actual waiting time of the n-th customer is then given by:

$$W_n^B = \sum_{k=0}^{K-1} u_n^k V_n^k, \qquad n \geq 0.$$

Developing recursively (14), and using the i.i.d. assumption to shift sums, these equations lead to the following "Loynes' Scheme":

$$\begin{aligned} W_0^B &= 0, \\ W_n^B &=_d \sum_{k=0}^{K-1} u_n^k \max_{0 \leq m \leq n-1} [\sum_{i=0}^{m} u_i^k \sigma_i - \sum_{i=1}^{m+1} \tau_i]^+ \qquad n \geq 1 \,. \end{aligned} \tag{15}$$

If $n < K$, $W_n^{RR} = 0$ so that relation (12) trivially holds. If $n \geq K$, set $n = Kp + K + l$ in (15) for some $p \geq 0$ and $0 \leq l < K$. Then,

$$\begin{aligned} W^B_{Kp+K+l} &=_d \sum_{k=0}^{K-1} u^k_{Kp+K+l} \max_{0\leq m\leq Kp+K+l-1} [\sum_{i=0}^{m} u_i^k \sigma_i - \sum_{i=1}^{m+1} \tau_i]^+ \\ &\geq_{st} \sum_{k=0}^{K-1} u^k_{Kp+K+l} \max_{0\leq n\leq Kp+K-1} [\sum_{i=0}^{n} u_i^k \sigma_i - \sum_{i=1}^{n+1} \tau_i]^+ \end{aligned}$$

by restricting the range of the max,

$$\geq_{st} \sum_{k=0}^{K-1} u^k_{Kp+K+l} \max_{0\leq m\leq p} [\sum_{i=0}^{Km+K-1} u_i^k \sigma_i - \sum_{i=1}^{Km+K} \tau_i]^+$$

by further restricting the range of the max to n's of the form Km,

$$=_d \sum_{k=0}^{K-1} u^k_{Kp+K+l} \max_{0\leq m\leq p} [\sum_{i=0}^{m} \tilde{\sigma}_i^k - \sum_{i=1}^{Km+K} \tau_i]^+ \tag{16}$$

where we set:

$$\tilde{\sigma}_i^k = \sum_{j=Ki}^{Ki+K-1} u_j^k \sigma_j .$$

For all k, the sequence $\{\tilde{\sigma}_n^k\}_0^\infty$ is an i.i.d. sequence. Moreover, we have that for all k and i:

$$\tilde{\sigma}_i^k =_d \sum_{j=1}^{\alpha} \sigma_j , \quad \text{where} \quad P(\alpha = j) = \binom{K}{j} \left(\frac{1}{K}\right)^j \left(1 - \frac{1}{K}\right)^{K-j} . \tag{17}$$

Since u_n^k is independent of u_m^j, $m < n$, we can assume, with no loss of generality, that $u_{Kp+K+l} = 1$. Consequently, inequality (16) reads

$$W^B_{Kp+K+l} \geq_{st} \max_{0\leq m\leq p} [\sum_{i=0}^{m} \tilde{\sigma}_i^1 - \sum_{i=1}^{Km+K} \tau_i]^+ . \tag{18}$$

Now, $1 = E\alpha \leq_{cx} \alpha$, therefore by Proposition 3.6, we have $\sigma_i \leq_{cx} \tilde{\sigma}_i^k$ for all k, i.

Consider now a $GI/GI/1$ system with i.i.d. interarrivals $\{\tau'_n\}_0^\infty$ defined by (13) and i.i.d. services $\{\tilde{\sigma}_n\}_0^\infty$, where $\tilde{\sigma}_n =_d \tilde{\sigma}_1^1$, $\forall n \geq 0$. Let $\{\overline{W}_n\}_0^\infty$ be the sequence of successive waiting times in this system. One recognizes on the right-hand side of (18) the Loynes' Scheme associated with this system:

$$\overline{W}_{Kp+K+l} =_d \max_{0\leq m\leq p} [\sum_{i=0}^{m} \tilde{\sigma}_i^1 - \sum_{i=1}^{Km+K} \tau_i]^+ .$$

Therefore, (18) should be read as:

$$W^B_{Kp+K+l} \geq_{st} \overline{W}_{Kp+K+l} .$$

But from standard comparison results, $\sigma_n \leq_{cx} \tilde{\sigma}_n$ implies $W_n^{RR} \leq_{icx} \overline{W}_n$ for all n. Finally,

$$W^{RR}_{Kp+K+l} \leq_{icx} W^B_{Kp+K+l}, \qquad p \geq 0,\ 0 \leq l < K .$$

□

6.5 Cycle Time of a Cyclic Polling System

Consider a polling system consisting of a single server and $K \geq 1$ queues with infinite capacity, numbered by $1, 2, \cdots, K$. The server visits the stations in a cyclic order, that is, $1, 2, \cdots, K, 1, 2, \cdots$. Without loss of generality, we assume that the server is at queue 1 at time zero. The initial state of the polling system is described by the vector $\mathbf{V} = (V_1, \cdots, V_K)$, where V_i is the (independent random) number of customers present at station i at time zero, $1 \leq i \leq K$. Customers arrive at the queues according to Poisson processes with parameters $\lambda_1, \cdots, \lambda_K$, respectively. The n-th customer ($n \geq 1$) of queue i requires $\sigma_n^i > 0$ service time. For every $i = 1, \cdots, K$, the random variables $\sigma_1^i, \cdots, \sigma_n^i, \cdots$, are i.i.d.. The sequences $\{\sigma_n^i\}_{n=1}^{\infty}$ are mutually independent. We first assume that there is no switch-over delay. At the n-th visit of the server to a queue, the server serves all the customers that were present at this queue at the beginning of its last visit to queue 1. This service discipline is introduced by Boxma et al. [8] and is called the "Globally Gated" discipline.

For $n \geq 1$, the time spent by the server at its n-th visit to queue i is referred to as the n-th station time at queue i, denoted by S_n^i. The n-th cycle time is defined to be

$$C_n = S_n^1 + \cdots + S_n^N.$$

Proposition 6.8 *Consider two globally gated cyclic polling systems whose initial states, customer arrival rates and service times are specified by* $\mathbf{V}^{(j)} = (V_1^{(j)}, \cdots, V_K^{(j)})$, $\{\lambda_1^{(j)}, \cdots, \lambda_K^{(j)}\}$, *and* $\{\sigma_n^{i,(j)},\ n \geq 1,\ 1 \leq i \leq K\}$, *for* $j = 1, 2$, *respectively. Let* $\leq_{\mathcal{L}} \in \{\leq_{\mathrm{st}}, \leq_{\mathrm{cx}}, \leq_{\mathrm{cv}}, \leq_{\mathrm{icx}}, \leq_{\mathrm{icv}}\}$. *If*

$$\lambda_i^{(1)} = \lambda_i^{(2)}, \qquad i = 1, \cdots, K,$$

and

$$V_i^{(1)} \leq_{\mathcal{L}} V_i^{(2)}, \quad \sigma_n^{i,(1)} \leq_{\mathcal{L}} \sigma_n^{i,(2)}, \qquad \forall n, i, \quad n \geq 1,\ 1 \leq i \leq K, \tag{19}$$

then

$$S_n^{i,(1)} \leq_{\mathcal{L}} S_n^{i,(2)}, \qquad n \geq 1, \quad 1 \leq i \leq K, \tag{20}$$

$$C_n^{(1)} \leq_{\mathcal{L}} C_n^{(2)}, \qquad n \geq 1, \quad 1 \leq i \leq K. \tag{21}$$

Proof It follows from the assumption (19) and Propositions 3.2—3.6 that

$$\forall i, \ 1 \leq i \leq K, \qquad S_1^{i,(1)} \leq_{\mathcal{L}} S_1^{i,(2)},$$

so that

$$C_1^{(1)} \leq_{\mathcal{L}} C_1^{(2)}.$$

Suppose that relations (20,21) hold for some $n \geq 1$. For system j, $j = 1, 2$, denote by $N_n^{i,(j)}$ the number of arrivals in queue i during the time interval $]X^{(j)}, X^{(j)} + C_n^{(j)}]$, where

$X^{(j)} = C_1^{(j)} + \cdots + C_{n-1}^{(j)}$. Since the arrivals are Poisson processes, $N_n^{i,(j)}$ is equivalent in law to the number of arrivals in queue i during the time interval $]0, C_n^{(j)}]$. Owing to Propositions 5.1 and 5.2, we have

$$\forall i, \ 1 \le i \le K, \qquad N_n^{i,(1)} \le_{\mathcal{L}} N_n^{i,(2)}.$$

Appealing again to Propositions 3.2—3.6 entails that

$$\forall i, \ 1 \le i \le K, \qquad S_{n+1}^{i,(1)} \le_{\mathcal{L}} S_{n+1}^{i,(2)},$$

so that

$$C_{n+1}^{(1)} \le_{\mathcal{L}} C_{n+1}^{(2)},$$

which completes the proof of the assertion by induction.

□

Remark: In an analogous way, one can show that if

$$(\lambda_1^{(1)}, \cdots, \lambda_K^{(1)}) \le (\lambda_1^{(2)}, \cdots, \lambda_K^{(2)})$$

componentwise, and

$$\mathbf{V}^{(1)} =_d \mathbf{V}^{(2)}, \qquad \sigma_n^{i,(1)} =_d \sigma_n^{i,(2)}, \quad \forall n, i, \quad n \ge 1, \ 1 \le i \le K,$$

then, for the globally gated and globally exhaustive systems,

$$S_n^{i,(1)} \le_{st} S_n^{i,(2)}, \qquad n \ge 1, \quad 1 \le i \le K\},$$

$$C_n^{(1)} \le_{st} C_n^{(2)}. \qquad n \ge 1.$$

Remark: All these comparison results for the polling systems can easily be extended to the case where switch-over times are incurred when the server switches from one queue to another. In this case, the station time is defined to be the time interval between the arrival instants of the server at two neighboring queues. The stochastic orderings $\le_{st}, \le_{cx}, \le_{cv}, \le_{icx}, \le_{icv}$ still hold when the switch-over times are thus ordered.

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References

[1] F. Baccelli, P. Brémaud, *Elements of Queueing Theory*, To appear in Applications of Mathematics, Springer Verlag, 1991.

[2] F. Baccelli, P. Brémaud, *Palm Probabilities and Stationary queues.* Lecture Notes in Statistics No. 41, Springer Verlag, 1987.

[3] F. Baccelli, Z. Liu, "On the Executions of Parallel Programs on Multiprocessor Systems— A Queueing Theory Approach". *Journal of the Association for Computing Machinery*, Vol. 37, 1990.

[4] F. Baccelli, A. Jean-Marie, Private communication.

[5] F. Baccelli, "Stochastic Ordering of Random Processes with an Imbedded Point Process", *INRIA Report No 1289, to appear in J. App. Prob., 1991.*

[6] R. Barlow, F. Proschan, *Statistical Theory of Reliability and Life Testing.* Holt, Rinehart and Winston, 1975.

[7] A. A. Borovkov, *Stochastic Processes in Queueing Theory*, Springer-Verlag, 1976.

[8] O. J. Boxma, H. Levy, U. Yechiali, "Cyclic Reservation Schemes for Efficient Operation of Multiple-Queue Single Server Systems", to appear in *Annals of Operations Research.*

[9] M. Brown, "Bounds, Inequalities and Monotonicity Properties for some Specialized Renewal Processes", *Ann. Prob. 8-2 pp. 227–240, 1980.*

[10] A. Erdélyi (Ed.), *Higher Transcendental Functions, vol. III*, McGraw Hill, 1955.

[11] W. Feller, *An Introduction to Probability Theory and its Applications* vol. 2, J. Wiley & Sons, 1966.

[12] L. Kleinrock, *Queueing Networks*, vol 1, J. Wiley & sons, 1975.

[13] T. Lindvall, "On Coupling of Renewal Processes with Use of Failure Rates", *Stoc. Proc. and Appl.*, 22, *pp. 1-15, 1986.*

[14] R. M. Loynes, "The Stability of Queues with Non Independent Inter-arrival and Service Times", *Proc. Cambridge Ph. Soc.* **58**, *pp. 497-520, 1962.*

[15] A. M. Makowski and T. K. Philips, "Simple Proofs for some Folk Theorems for Parallel Queues", *IBM Research Report* **RC-14556**, *1988*

[16] A. W. Marshall and I. Olkin, *Inequalities: Theory of Majorization and its Applications* Academic press, New York, 1979.

[17] T. Rolski and R. Szeckli, "Stochastic Ordering and Thinning of Point Processes", to appear in *Stoc. Proc. and Appl., 1991.*

[18] S. Ross, *Stochastic Processes*, J. Wiley & Sons, 1984.

[19] D. Stoyan, *Comparison Methods for Queues and Other Stochastic Models.* English translation (D.J. Daley editor), J. Wiley and Sons, New York, 1984.

[20] V. Strassen, "The Existence of Probability Measures with Given Marginals," *Ann. Math. Stat.*, Vol. 36, pp. 423-439, 1965.

[21] Tedijanto, "A Note on the Comparison of Limited and Bernoulli Policies in Vacation Models", *to appear in Performance Evaluation, 1991.*

[22] W. Whitt, "Comparing Counting Processes and Queues," *Adv. Appl. Prob.*, Vol. 13, pp. 207-220, 1981.

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